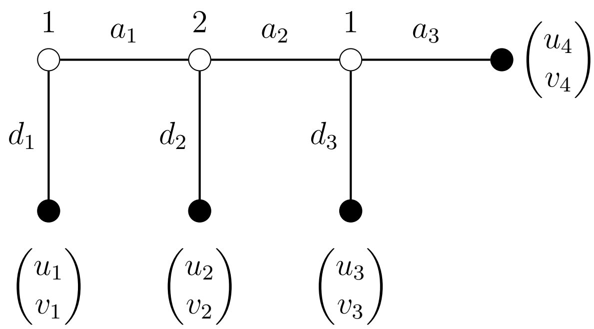
Problem Set on Representations, Domination, and Strategic Form Games

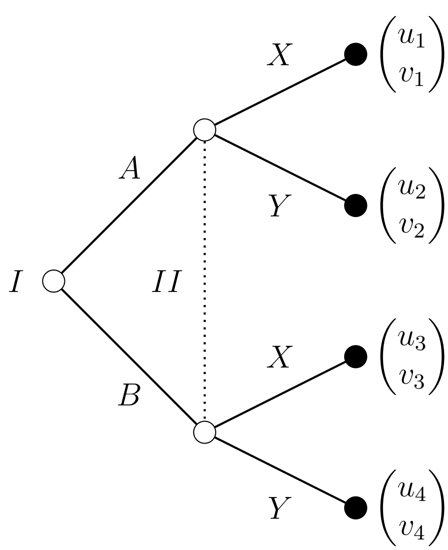
1. Describe the following situation as an extensive-form game. By drawing arrows on the game tree, determine if there is a winning player, or a guaranteed draw: On a table is a single match, a pile of two matches, and a pile of three matches. Two players take turns removing matches from the piles. At a player's turn she can remove one or more matches, but only from one pile. The player who removes the last match loses.

2. Find the strategic form for the following games.

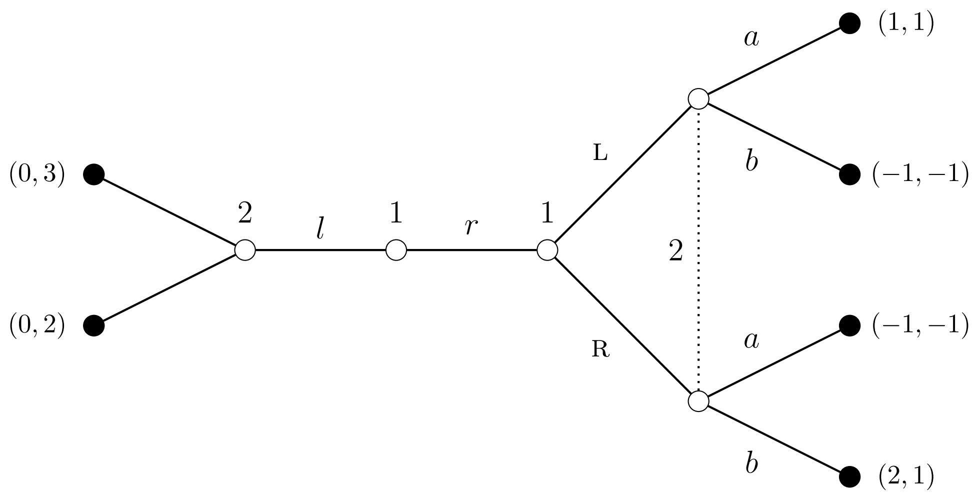
   (a)



   (b)



3. Consider this game:



(a) List all the strategies of the game for each player.

(b) Described a completely mixed behavior strategy for each player.

(c) Construct a mixed strategy that gives the same probability distribution over the terminal nodes.

(d) Take a different mixed strategy profile, and find the behavior strategy which gives the same probability over terminal nodes.

(e) Describe the normal form of this game.

4. Thank Joel Watson for this problem: Consider the following strategic setting involving a cat named Baker, a mouse named Cheezy, and a dog named Spike. Baker's objective is to catch Cheezy while avoiding Spike; Cheezy wants to tease Baker but avoid getting caught; Spike wants to rest and is unhappy when he is disturbed. In the morning, Baker and Cheezy simultaneously decide what activity to engage in. Baker can either nap (N) or hunt (H), where hunting involves moving Spike's bone. Cheezy can either hide (h) or play (p). If nap and hide are chosen, then the game ends. The game also will end immediately if hunt and play are chosen, in which case Baker captures Cheezy. On the other hand, if nap and play are chosen, then Cheezy observes that Baker is napping and must decide whether to move Spike's bone (m) or not (n). If he chooses to not move the bone, then the game ends. Finally, in the event that Spike's bone was moved (either by Baker choosing to hunt or by Cheezy moving it later), then Spike learns that his bone was moved but does not observe who moved it; in this contingency, Spike must choose whether to punish Baker (B) or punish Cheezy (J). After Spike moves, the game ends. In this game, how many information sets are there for Cheezy? How many strategy profiles are there in this game?

5. Another Watson novella: Suppose we have a game where $S_1 = \{H, L\}$ and $S_2 = \{X, Y\}$. If player 1 plays H, then her payoff is z regardless of player 2's choice of strategy; player

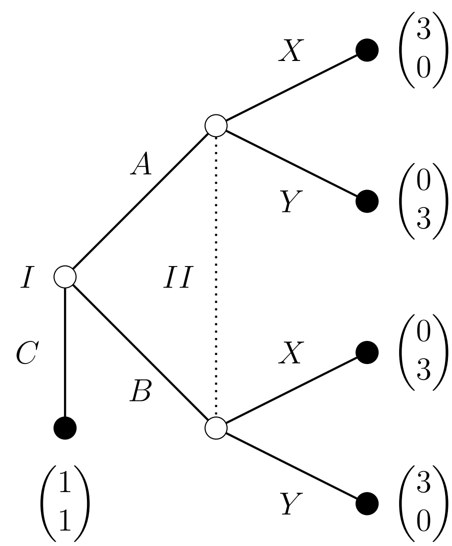
1's other payoff numbers are $u_1(L, X) = 0$ and $u_1(L, Y) = 10$. You may choose any payoff numbers you like for player 2 because we will only be concerned with player l's payoff.

(a) Draw the normal form of this game.

(b) If player 1's belief about 2's strategy choice is $\theta_2 = (1/2, 1/2)$, what is player 1's expected payoff of playing H? What is the expected payoff of playing L? For what value of z is player 1 indifferent between playing H and L?

(c) Suppose $\theta_2 = (1/3, 2/3)$. Find player 1's expected payoff of playing L.

6. In the following game, is C a best response?



7. Candidates A and B are running for office. Each of an odd number of people will vore for one or the other candidate — no abstentions. The winner is chosen by majority rule, each voter cares only about who wins, not the margin of victory, and every voter has one favored candidate. Let us suppose that a majority of voters prefer A.

(a) Describe this situation as a normal form game.

(b) For each voter, describe the set of weakly dominated strategies.

(d) Suppose there are K candidates rather than 2, and each player has a strict preference order over candidates. Suppose that a voter who prefers a win by A to a win by Z ranks a tie in between the two. What is weakly dominated here?

8. Provide an example of a two-player game with strategy set $[0, \infty)$ for both player, and payoffs continuous in the strategy profile, such that no strategy survices the iterated elimination of strictly dominated strategies, but for which the set of strategies remaining at every stage is non-empty.

9. In the normal form below, player 1 chooses a row, 2 a column, and 3 a matrix. The payoffs below are only for player 3.

	L	R
U	9	0
D	0	0

A

	L	R
U	0	9
D	9	0

B

	L	R
U	0	0
D	0	9

C

	L	R
U	6	0
D	0	6

D

(a) Is D strictly dominated in the mixed extension?

(b) Is D ever a best response?

10. Consider the following game:

	A_c	B_c	C_c
A_r	10,0	0,10	3,5
B_r	0,10	10,0	3,5
C_r	5,3	5,3	2,2

(a) Find ALL the Nash equilibria of this game. What are their expected payoffs?

(b) Which equilibria are admissible?

(c) Suppose players take turns eliminating weakly dominated strategies, starting with player 1. What is the outcome? Is it admissible?

11. Find all equilibria of

	L	R
U	0,0	3,1
D	1,3	2,2

	L	C	R
T	0,2	3,3	1,1
M	4,5	0,4	2,0
B	0,4	5,1	0,7

	B	S	X
B	4,2	0,0	0,1
S	0,0	2,4	1,3

12. Consider the following two-player game:

	W	X	Y	Z
A	4,2	0,0	5,0	0,0
B	1,4	1,4	0,5	1,0
C	0,0	2,4	1,2	0,0
D	0,0	0,0	0,1	0,0

(a) What game results from iteratively deleting strictly dominated strategies?

(b) Explain the assumptions on rationality and (higher-order) knowledge of rationality for each elimination.

13. Find all (pure) equilibria of the following two-player game:

	A	B	C
X	3,1	0,0	1,0
Y	0,0	1,3	1,1
Z	1,1	0,1	0,10

14. Consider the games

	L	R
U	a, b	0,0
D	0,0	1,1

with $-2 < a, b < 2$.

(a) For various values of b, plot how the equilibrium probability of U varies with a. Be sure to plot the variation for all equilibria.

(b) For various values of a, plot how the equilibrium probability of U varies with b. Be sure to plot the variation for all equilibria.

15. Who reports a crime? N individuals witness a crime. Each player can either *call* the police, or *not*. Preferences are: utility 0 if no one calls, $v > 0$ if someone else calls, and $v - c > 0$, where c is the cost of having to make the call.

(a) Find all the pure equilibria.

(b) Find the symmetric mixed equilibrium.

(c) In the symmetric mixed equilibrium, how does the probability that *someone* makes a call vary with N, the number of players?

16. Compute the Nash equilibria in the following linear Cournot oligopoly games, assuming that all firms simultaneously make their production decisions.

(a) There are N firms. Each firm has marginal cost $c > 0$ and no fixed cost. The inverse demand is $P(Q) = \max\{0, 1 - Q\}$ where Q is total supply.

(b) There are a potentially unlimited number of entrants. Each firm must bear a fixed cost $F > 0$ if and only if it produces.

17. Two individuals, 1 and 2, contribute to a public good (say, a clean shared kitchen) by making individual efforts $x \in [0, 1]$ and $y \in [0, 1]$. The resulting level of the public good is $x + y$. Individual utilities are given by

$$u_1(x, y) = (x + y)e^{-x} \text{ and } u_2(x, y) = (x + y)e^{-y}.$$

Each individual strives to maximize his or her expected utility.

(a) Game A: Suppose both effort levels are chosen simultaneously. Write up the normal form of Game A. Find all the pure-strategy Nash equilibria.

(b) Game B: Suppose individual 1 first chooses her effort level, and that this is observed by individual 2, who then chooses his effort level. Describe the normal form of Game B, and find all the subgame perfect equilibria.

(c) Does there exist a Nash equilibrium in Game B in which individual 2 makes effort level $y = 1/2$? (Either prove than none exists or specify one such equilibrium.)

18. A population of individuals $\mathcal{I} = \{1, \ldots, I\}$ are asked to contribute to a public good. Preferences for individual i are $u_i(c_i, g) = c_i + \alpha_i \ln g$ where c is the quantity of consumption good consumed by individual i and g is the quantity of public good supplied. Each individual chooses a "gift" g_i, and $g = \sum_i g_i$. Individual i has wealth w_i, and her gift and consumption must satisfy the constraint $c_i + g_i \leq w_i$.

    (a) Suppose that all consumers have identical wealth $wi = w$, and that α_i is increasing in i. Describe the possible Nash equilibria that could arise.

    (b) Suppose that all the taste parameters are identical, $\alpha_i = \alpha$, and that the w_i are increasing in i. Describe the possible Nash equilibria that could arise.

    (c) Show that for any distribution of tastes and wealth, the Nash equilibrium is unique.

    (d) Suppose two individuals, say 1 and 2, are giving in a particular Nash equilibrium, and that a tiny amount ϵ of wealth is transferred from individual 1 to individual 2. Describe the new equilibrium. What happens to the provided level of public good?

    (e) Suppose two individuals, say 1 and 2, are not giving in a particular Nash equilibrium, and that a tiny amount ϵ of wealth is transferred from individual 1 to individual 2. Describe the new equilibrium. What happens to the provided level of public good?

    (f) Suppose that individual 1 is a giver and individual 2 is not. What is the effect of a wealth transfer from 2 to 1?

19. Consider the two-player game

	L	R
T	$2, 7$	$0, 0$
B	$2, 0$	$3, 1$

    (a) Find all the pure and mixed Nash equilibria

    (b) Which Nash equilibria (if any) are un-weakly dominated.

20. Consider two competing firms in a declining industry that cannot support both firms profitably. Each firm has three possible choices, as it must decide whether or not to exit the industry immediately, at the end of this quarter, or at the end of the next quarter. If a firm chooses to exit then its payoff is 0 from that point onward. Each quarter that both firms operate yields each a loss equal to -1, and each quarter that a firm operates alone yields it a payoff of 2. For example, if firm 1 plans to exit at the end of this quarter while firm 2 plans to exit at the end of the next quarter then the payoffs are $(-1, 1)$ because both firms lose -1 in the first quarter and firm 2 gains 2 in the second. The payoff for each firm is the sum of its quarterly payoffs.

(a) Write down this game in matrix form.

(b) Are there any strictly dominated strategies? Are there any weakly dominated strategies?

(c) Find the pure-strategy Nash equilibria.

(d) Find the unique mixed-strategy Nash equilibrium.

21. Give an example of a two-player game with strategy sets $\mathbf{R}_+$ and with continuous payoff functions for which the set of iteratively un-weakly dominated strategies is empty.

22. Find a game that has admissible equilibria which are not trembling-hand perfect. Hint: How many players do you need?

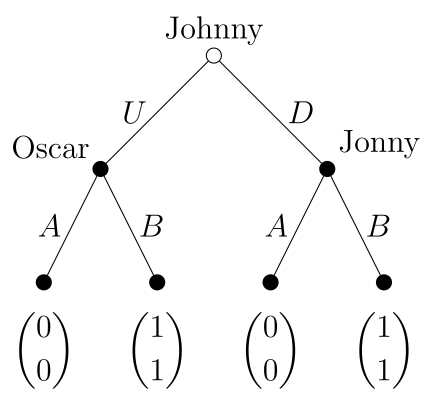
23. A Nash equilibrium is strict iff each player's strategy is the unique best response to the play of others.

    (a) Show that any strict Nash equilibrium can use only pure strategies.

    (b) Show that every strict Nash equilibrium is trembling-hand perfect.

    (c) Give an example of a game and a trembling-hand perfect Nash equilibrium that is not strict.

24. Consider this game:



    (a) Find all the pure SPE.

    (b) Find all the extensive-form perfect equilibria.

    (c) Write down the normal form, and find all the trembling-hand perfect equilibria.

    The conclusion you should reach from comparing the answers to the three parts is a bit surprising.