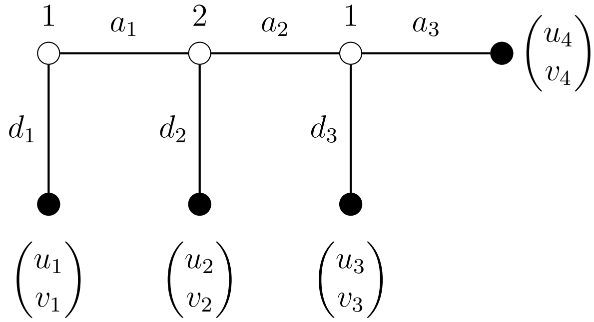
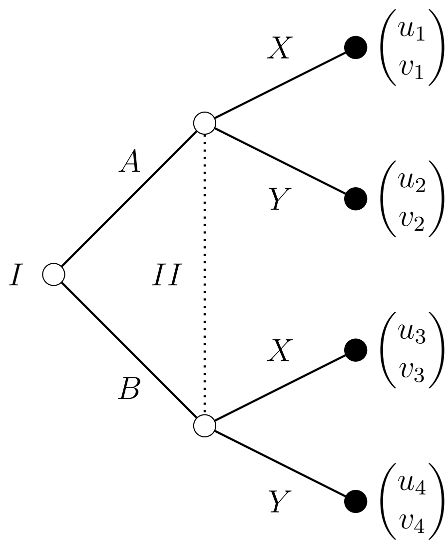
Game Theory Problems

1. Find the strategic form for the following games.

   (a)



   (b)



Solution:

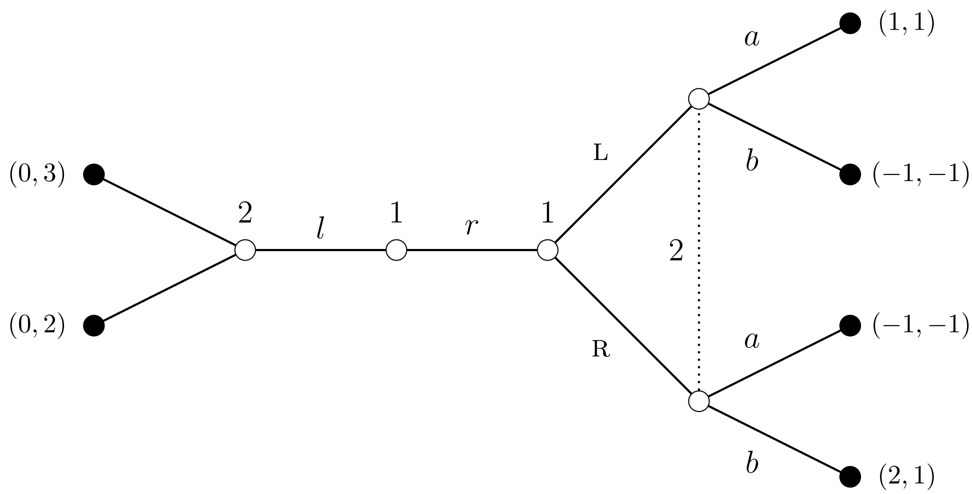
a)

	a_2	d_2
a_1a_3	u_4v_4	u_2v_2
a_1d_3	u_3v_3	u_2v_2
d_1a_3	u_1v_1	u_1v_1
d_1d_3	u_1v_1	u_1v_1

b)

	X	Y
A	u_1v_1	u_2v_2
B	u_3v_3	u_4v_4

2. Consider this game:



(a) List all the strategies of the game for each player.

(b) Described a completely mixed behavior strategy for each player.

(c) Construct a mixed strategy that gives the same probability distribution over the terminal nodes.

(d) Take a different mixed strategy profile, and find the behavior strategy which gives the same probability over terminal nodes.

(e) Describe the normal form of this game.

3. Thank Joel Watson for this problem: Consider the following strategic setting involving a cat named Baker, a mouse named Cheezy, and a dog named Spike. Baker's objective is to catch Cheezy while avoiding Spike; Cheezy wants to tease Baker but avoid getting caught; Spike wants to rest and is unhappy when he is disturbed. In the morning, Baker and Cheezy simultaneously decide what activity to engage in. Baker can either nap (N) or hunt (H), where hunting involves moving Spike's bone. Cheezy can either hide (h) or play (p). If nap and hide are chosen, then the game ends. The game also will end immediately if hunt and play are chosen, in which case Baker captures Cheezy. On the other hand, if nap and play are chosen, then Cheezy observes that Baker is napping and must decide whether to move Spike's bone (m) or not (n). If he chooses to not move the bone, then the game ends. Finally, in the event that Spike's bone was moved (either by Baker choosing to hunt or by Cheezy moving it later), then Spike learns that his bone was moved but does not observe who moved it; in this contingency, Spike must choose whether to punish Baker (B) or punish Cheezy (J). After Spike moves, the game ends. In this game, how many information sets are there for Cheezy? How many strategy profiles are there in this game?

Solution: One has to draw the game tree. Cheezy has 2 information sets. $|S_B| = |S_S| = 2$ and $|S_C| = 4$, so there are $2 \times 2 \times 4 = 16$ pure strategy profiles.

4. Another Watson novella: Suppose we have a game where $S_1 = \{H, L\}$ and $S_2 = \{X, Y\}$. If player 1 plays H, then her payoff is z regardless of player 2's choice of strategy; player 1's other payoff numbers are $u_1(L, X) = 0$ and $u_1(L, Y) = 10$. You may choose any payoff numbers you like for player 2 because we will only be concerned with player l's payoff.

   (a) Draw the normal form of this game.

   Solution:

	X	Y
H	z, _	z, _
L	0, _	10, _

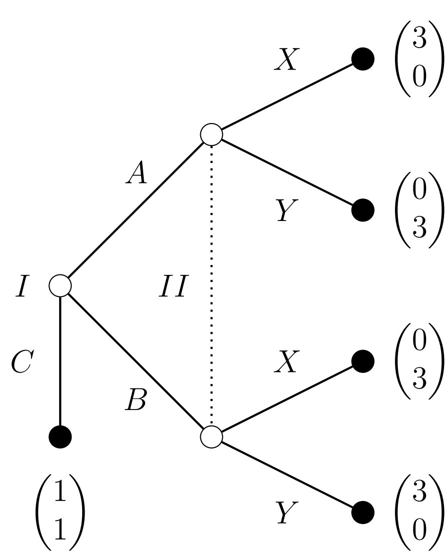
   (b) If player 1's belief about 2's strategy choice is $\theta_2 = (1/2, 1/2)$, what is player 1's expected payoff of playing H? What is the expected payoff of playing L? For what value of z is player 1 indifferent between playing H and L?

   Solution: Expected utilities are z and 5 respectively. 1 is indifferent between H and L when $z = 5$.

   (c) Suppose $\theta_2 = (1/3, 2/3)$. Find player 1's expected payoff of playing L.

   Solution: 20/3.

5. In the following game, is C a best response?



Solution: Look at the normal form.

	X	Y
A	$3,0$	$0,3$
B	$0,3$	$3,0$
C	1,1	1,1

C is undominated in pure strategies, but dominated in mixed strategies by $(1/2)A + (1/2)B$. Thus it is never a best reply.

6. Candidates A and B are running for office. Each of an odd number of people will vore for one or the other candidate — no abstentions. The winner is chosen by majority rule, each voter cares only about who wins, not the margin of victory, and every voter has one favored candidate. Let us suppose that a majority of voters prefer A.

   (a) Describe this situation as a normal form game.

   Solution: $G = \langle \mathcal{N}, (u_n, S_n)_{n\in\mathcal{N}} \rangle$ where $\mathcal{N} = \{1, \ldots, 2k+1\}$. Suppose that voters $1, \ldots, k+l$ prefer A to B, where $0 < l \leq k+1$.

   $$u_n(s) = \begin{cases} 1 & \text{if } \#\{n : s_n = A\} > k, \\ 0 & \text{otherwise.} \end{cases}$$

   (b) For each voter, describe the set of weakly dominated strategies.

   Solution: For voters who prefer A to B, anything other than voting for A is weakly dominated. For voters who prefer B to A, anything other than voting for B is weakly dominated. In both cases, one player's vote has no effect except in the event that there is a tie among all the other players. In that case, her vote determines the winner — she is *pivotal* — and in this case she is best off voting her preference.

   (c) Find all the Nash equilibria. Are there equilibria in which players use weakly dominated strategies?

   Solution: The Nash equilibria are:

   - Any $k+1$ A preferers vote for A, everyone else votes any arbitrary way.
   - Any $k+2$ voters vote for A, everyone else votes any arbitrary way.
   - Any $k+2$ voters vote for B, everyone else votes any arbitrary way.

   Each case contains equilibria in which some players are using weakly dominated strategies. Voting one"s true preferences is also an equilibrium.

   (d) Suppose there are K candidates rather than 2, and each player has a strict preference order over candidates. Suppose that a voter who prefers a win by A to a win by Z ranks a tie in between the two. What is weakly dominated here?

Solution: The only weakly dominated strategy is bidding one's last choice. In all other cases, a vote for candidate X over a voter's last choice Z guarantees a win for X over Z in the event that there is a tie for top choice between X and Z among the remaining voters.

7. Provide an example of a two-player game with strategy set $[0, \infty)$ for both player, and payoffs continuous in the strategy profile, such that no strategy services the iterated elimination of strictly dominated strategies, but for which the set of strategies remaining at every stage is non-empty.

Solution: Take $u_i(x_i, x_j) = -|x_i - (x_j + 1)|$, or (equivalently) $u_i(x_i, x_j) = -\big(x_i - (x_j + 1)\big)^2$.

8. In the normal form below, player 1 chooses a row, 2 a column, and 3 a matrix. The payoffs below are only for player 3.

	L	R
U	9	0
D	0	0

A

	L	R
U	0	9
D	9	0

B

	L	R
U	0	0
D	0	9

C

	L	R
U	6	0
D	0	6

D

(a) Is D strictly dominated in the mixed extension?

Solution: No. D is a best response to the probability distribution that puts mass 1/2 on each of UL and DR. Since it is a best response to some probability distribution, it cannot be dominated.

(b) Is D ever a best response?

Solution: No. A somewhat tedious check shows that against any product distribution on rows and columns, D is beat by one or more of A, B and C.

9. Consider the following game:

	A_c	B_c	C_c
A_r	10,0	0,10	3,5
B_r	0,10	10,0	3,5
C_r	5,3	5,3	2,2

(a) Find ALL the Nash equilibria of this game. What are their expected payoffs?

Solution: Both players playing A and C each with probability 1/2 is an admissible equilibrium. The expected payoff to each player is 5. There is also a pair of equilibria where one player chooses C and the other player used a mixed strategy with probability 1/2 on each of A and B. These are inadmissible since C is weakly dominated by the 1/2 each mixture of A and B. Finally, there are a continua of inadmissible equilibria where one player chooses $(1/2, 1/2, 0)$ and the other chooses $(p/2, p/2, 1 - p)$ for $0 < p < 1$.

(b) Which equilibria are admissible?

(c) Suppose players take turns eliminating weakly dominated strategies, starting with player 1. What is the outcome? Is it admissible?

Solution: If player 1's C_r is eliminated, C_r is no longer weakly dominated. The equilibria wherein player 1 plays $(1/2, 1/2, 0)$ and player 2 plays $(p/2, p/2, 1-p)$ still remain, and none are weakly dominated in the reduced game.

10. Find all equilibria of

	L	R
U	0,0	3,1
D	1,3	2,2

	L	C	R
T	0,2	3,3	1,1
M	4,5	0,4	2,0
B	0,4	5,1	0,7

	B	S	X
B	4,2	0,0	0,1
S	0,0	2,4	1,3

Solution: For the 3×3 game there is TL and a completely mixed equilibrium. Also $(3/4, 0, 1/4)$ and $(0, 1/3, 2/3)$ for row and col, respectively. For the 2×3 game, BB and SS are equilibria. Col plays $(1/3, 2/3, 0)$ and row plays $(2/3, 1/3)$ is not an equilibrium since the payoff to X is $5/3 > 4/3$, the payoff to B and S.

11. Consider the following two-player game:

	W	X	Y	Z
A	4,2	0,0	5,0	0,0
B	1,4	1,4	0,5	1,0
C	0,0	2,4	1,2	0,0
D	0,0	0,0	0,1	0,0

(a) What game results from iteratively deleting strictly dominated strategies?

Solution: Row has strategies A, C and Column has strategies W, X. Some eliminated strategies are dominated only by mixed strategies.

(b) Explain the assumptions on rationality and (higher-order) knowledge of rationality for each elimination.

Solution: Row and Column are both rational, so Row will not play D and Column will not play D. Row knows that Column is rational, so he knows that Column will not play Z. Since he is rational, he will not play B. Column knows that Row knows that Column is rational, so she knows that Row knows that she will not play Z. She knows too that Row is rational, so she knows that Row will not play B. Since she is rational, she will not play Y.

12. Find all (pure) equilibria of the following two-player game:

	A	B	C
X	3,1	0,0	1,0
Y	0,0	1,3	1,1
Z	1,1	0,1	0,10

Solution: The pure NE are XA and YB.

13. Consider the games

	L	R
U	a, b	0,0
D	0,0	1,1

with $-2 < a, b < 2$.

(a) For various values of b, plot how the equilibrium probability of U varies with a. Be sure to plot the variation for all equilibria.

(b) For various values of a, plot how the equilibrium probability of U varies with b. Be sure to plot the variation for all equilibria.

Solution: The main thing to notice is that in the mixed equilibria, changing columns payoffs affects how row plays, but not how column plays. And similarly for row.

14. Who reports a crime? N individuals witness a crime. Each player can either *call* the police, or *not*. Preferences are: utility 0 if no one calls, $v > 0$ if someone else calls, and $v - c > 0$, where c is the cost of having to make the call.

(a) Find all the pure equilibria.

(b) Find the symmetric mixed equilibrium.

Solution: If a player calls, his utility is $v-c$. If he does not call and the probability of everyone else calling is q, the probability of no one calling is $(1-q)^{n-1}$. Thus the expected payoff to not calling is $\{1-(1-q)^{n-1}\}v$. In a mixed equilibrium, these two payoffs must be equal, so $1-q = (c/v)^{1/n-1}$.

(c) In the symmetric mixed equilibrium, how does the probability that *someone* makes a call vary with N, the number of players?

Solution: The probability that no one calls is $(c/v)^{n/n-1}$ which decreases with n.

15. Compute the Nash equilibria in the following linear Cournot oligopoly games, assuming that all firms simultaneously make their production decisions.

(a) There are N firms. Each firm has marginal cost $c > 0$ and no fixed cost. The inverse demand is $P(Q) = \max\{0, 1 - Q\}$ where Q is total supply.

(b) There are a potentially unlimited number of entrants. Each firm must bear a fixed cost $F > 0$ if and only if it produces.

16. Two individuals, 1 and 2, contribute to a public good (say, a clean shared kitchen) by making individual efforts $x \in [0, 1]$ and $y \in [0, 1]$. The resulting level of the public good is $x + y$. Individual utilities are given by

$$u_1(x, y) = (x + y)e^{-x} \text{ and } u_2(x, y) = (x + y)e^{-y}.$$

Each individual strives to maximize his or her expected utility.

(a) Game A: Suppose both effort levels are chosen simultaneously. Write up the normal form of Game A. Find all the pure-strategy Nash equilibria.

Solution: Any pair (x, y) such that $x + y = 1$ is a Nash equilibrium.

(b) Game B: Suppose individual 1 first chooses her effort level, and that this is observed by individual 2, who then chooses his effort level. Describe the normal form of Game B, and find all the subgame perfect equilibria.

Solution: The game has $\mathcal{N} = \{1, 2\}$. $S_1 = [0, 1]$. $S_2 = \{f : [0, 1] \to [0, 1]\}$. In an SPE player 2 must respond optimally to every x, so his strategy is $f(x) = 1 - x$. The optimal strategy for 1 is then to choose $x = 0$, so $(0, 1)$ is the only SPE outcome.

(c) Does there exist a Nash equilibrium in Game B in which individual 2 makes effort level $y = 1/2$? (Either prove than none exists or specify one such equilibrium.)

Solution: f(x) = 1/2 if $x = 1/2$ and 0 otherwise, and $x = 1/2$.

17. A population of individuals $\mathcal{I} = \{1, \ldots, I\}$ are asked to contribute to a public good. Preferences for individual i are $u_i(c_i, g) = c_i + \alpha_i \ln g$ where c is the quantity of consumption good consumed by individual i and g is the quantity of public good supplied. Each individual chooses a "gift" g_i, and $g = \sum_i g_i$. Individual i has wealth w_i, and her gift and consumption must satisfy the constraint $c_i + g_i \leq w_i$.

(a) Suppose that all consumers have identical wealth $wi = w$, and that α_i is increasing in i. Describe the possible Nash equilibria that could arise.

(b) Suppose that all the taste parameters are identical, $\alpha_i = \alpha$, and that the w_i are increasing in i. Describe the possible Nash equilibria that could arise.

(c) Show that for any distribution of tastes and wealth, the Nash equilibrium is unique.

(d) Suppose two individuals, say 1 and 2, are giving in a particular Nash equilibrium, and that a tiny amount ϵ of wealth is transferred from individual 1 to individual 2. Describe the new equilibrium. What happens to the provided level of public good?

(e) Suppose two individuals, say 1 and 2, are not giving in a particular Nash equilibrium, and that a tiny amount ϵ of wealth is transferred from individual 1 to individual 2. Describe the new equilibrium. What happens to the provided level of public good?

(f) Suppose that individual 1 is a giver and individual 2 is not. What is the effect of a wealth transfer from 2 to 1?

18. Consider the two-player game

	L	R
T	$2,7$	$0,0$
B	$2,0$	$3,1$

(a) Find all the pure and mixed Nash equilibria

(b) Which Nash equilibria (if any) are un-weakly dominated.

19. Consider two competing firms in a declining industry that cannot support both firms profitably. Each firm has three possible choices, as it must decide whether or not to exit the industry immediately, at the end of this quarter, or at the end of the next quarter. If a firm chooses to exit then its payoff is 0 from that point onward. Each quarter that both firms operate yields each a loss equal to -1, and each quarter that a firm operates alone yields it a payoff of 2. For example, if firm 1 plans to exit at the end of this quarter while firm 2 plans to exit at the end of the next quarter then the payoffs are $(-1, 1)$ because both firms lose -1 in the first quarter and firm 2 gains 2 in the second. The payoff for each firm is the sum of its quarterly payoffs.

(a) Write down this game in matrix form.

(b) Are there any strictly dominated strategies? Are there any weakly dominated strategies?

(c) Find the pure-strategy Nash equilibria.

(d) Find the unique mixed-strategy Nash equilibrium.

20. Give an example of a two-player game with strategy sets $\mathbf{R}_+$ and with continuous payoff functions for which the set of iteratively un-weakly dominated strategies is empty.

Solution: $\mathcal{N} = \{1, 2\}$, $S_1, S_2 = (0, 1]$, $u_i(s_i, s_{-i}) = \min\{s_i, s_{-i} - s_i\}$.

21. Find a game that has admissible equilibria which are not trembling-hand perfect. Hint: How many players do you need?

Solution: Players 1 2 and 3 choose row, column and matrix, respectively.

	L	R
U	1,1,1	1,1,1
D	1,1,1	0,0,0

A

	L	R
U	1,1,1	0,0,0
D	0,0,0	1,1,1

B

U, L, B is an admissible Nash equilibrium. It is not trembling-hand perfect since against any sequence of trembles by player 1 and 2 converging to U and L, respectively, B is not a best response.

22. A Nash equilibrium is strict iff each player's strategy is the unique best response to the play of others.

    (a) Show that any strict Nash equilibrium can use only pure strategies.

    Solution: If an equilibrium strategy profile contains a mixed (impure) strategy, then all pure strategies in the support of that mixed strategy have to be best responses against the profile of the other players; hence the equilibrium is not strict.

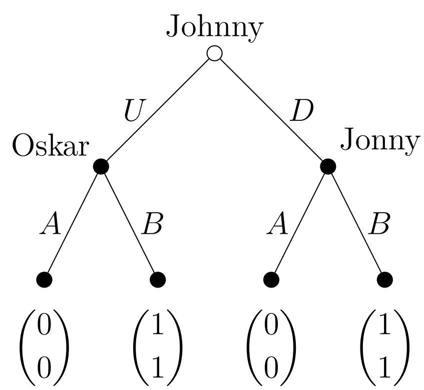
    (b) Show that every strict Nash equilibrium is trembling-hand perfect.

    Solution: Let $s = (s_1, \ldots, s_n)$ denote a strict NE, and let $\sigma = (\sigma_1^k, \ldots, \sigma_n^k)$ denote a sequence of completely mixed strategies converging to $(s_1, \ldots, s_n)$. For player i consider alternative t_i. Since s_i is better than t_i against s_{-i}, it will be better against σ_{-i}^k for large enough k.

    (c) Give an example of a game and a trembling-hand perfect Nash equilibrium that is not strict.

    Solution: The unique Nash equilibrium of matching pennies.

23. Consider this game:



    (a) Find all the pure SPE.

    Solution: U, B and A, B are both SPE's.

    (b) Find all the extensive-form perfect equilibria.

    Solution: All SPE's are extensive-form perfect equilibria. To see why look at the agent normal form and ask what happens if Johnny's trembles at his second move are larger than those at his first.

    (c) Write down the normal form, and find all the trembling-hand perfect equilibria.

    Solution:

		Johnny			
		UA	UB	DA	DB
Oskar	A	$0,0$	$0,0$	$0,0$	$1,1$
	B	$1,1$	$1,1$	$0,0$	$1,1$

DB is Johnny's only admissible strategy. Since a perfect equilibrium must be admissible, and since one must exist, Johnny must use DB in it. A is also inadmissible for Oskar, the only perfect equilibrium is DB, B.

The conclusion you should reach from comparing the answers to the three parts is a bit surprising.

24. $N = \{I, II\}$. Player I has two types: $T_I = \{I_1, I_2\}$. Player II has only one type, $T_{II} = \{II\}$. The payoff matrices are:

	L	R
T_1	$1,0$	$0,2$
B_1	$0,3$	$1,0$

I_1, II

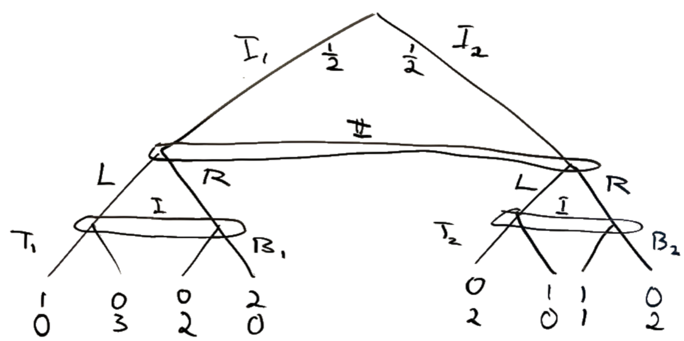
	L	R
T_2	$0,2$	$1,1$
B_2	$1,0$	$0,2$

I_2, II

Suppose $p(I_1, II) = p(I_2, II) = 1/2$. Notice that in this game player 2 does not know her own payoff function. Let x denote the probability that type I_1 plays T_1, y denote the probability that type I_2 plays T_2, and z denote the probability that type II plays L,

(a) Draw an extensive form for this game.

Solution:



(b) Show that in a Bayes Nash equilibrium $0 < z < 1$.

Solution: Compute the normal form.

		I			
		T_1T_2	T_1B_2	B_1T_2	B_1B_2
II	L	$1/2, 1$	$1, 0$	$0, 5/2$	$0, 5/2$
	R	$1/2, 3/2$	$0, 2$	$0, 5/2$	$0, 5/2$

T_1T_2 is dominated by a mixture of T_1B_2 and any of the B_1X strategies that puts weight slightly more than $1/2$ on T_1B_2. After elimination, it is clear that there is no pure strategy equilibrium, and that L must be used.

(c) Find all the Bayes Nash equilibria.

Solution: Since the B_1X strategies are payoff-equivalent, there are a lot of them. R is played with probability $5/7$ and T_1B_2 with probability $1/2$.

25. A public good is provided to a group of N people if at least one of them is willing to pay the cost c of production. The value of the good to person n is v_n. Every individual puts either c or 0 into an envelope and submits it to an arbiter. If no envelope contains c, then the good is not provided and everyone gets utility 0. If one or more envelopes contains a c, the good is provided. Any n who put in 0 gets value v_n, and any individual who puts in c gets $v_n - c$.

    Every individual knows only her own v_n. The v_n are independently and identically distributed on an interval $[v', v'']$ with a cumulative distribution function F which is strictly increasing.

    (a) Set this up as a Bayesian game.

    (b) Find a symmetric Bayes Nash Equilibrium. Hint: Look for a cut-off strategy, where there is a value v^* such that those with values above v^* give and those with values below do not.

26. Here is a game in normal form:

		Col X	Col Y
Row	X	$3, 2$	$1, 1$
Row	Y	$4, 3$	$2, 4$

    (a) Find all the Nash equilibria of this game.

    (b) Now suppose that Row chooses first, followed by Col. Draw the extensive form.

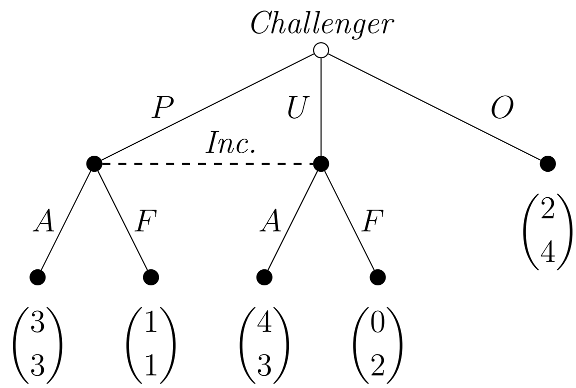
    (c) Find all the subgame perfect equilibria. How can it be that there are subgame perfect equilibria that you did not find in the preceding part?

    (d) Now suppose that Col only imperfectly observes Row's choice. With probability $1 - \epsilon$ Col sees the move Row made, and with probability Col sees the other choice, the choice Row did not make. Draw the extensive form and the normal for this Bayesian game.

    (e) Find all the pure-strategy Nash equilibria for this game.

    (f) Find the mixed equilibria for this game.

27. Here is an entry game. An *incumbent* faces the threat of entry by a *Challenger*. The challenger has three choices: It can Prepare for entry, enter quickly but Unprepared, or stay Out. The incumbent may either Fight or Acquiesce. Preparation is costly but reduces the loss from a fight. A fight is less costly to the incumbent if the challenger is unprepared, but regardless, the incumbent prefers to acquiesce.



(a) What is the normal form of this game?

(b) Is it plausible that the outcome of the game will be (O, F)?

(c) Describe the set of possible consistent beliefs for the incumbent.

(d) Is there a consistent and sequentially rational assessment with equilibrium strategy profile (O, F)?