CHAPTER NINE

Linear Models of Production

We have already devoted considerable space to the analysis of models of production in connection with linear programming problems. We were interested there in questions of maximizing revenues, minimizing costs, and the like. In this chapter we shall consider questions connected with production models which are not matters of simple optimization. Rather than attempt to describe in advance the material to be discussed let us proceed at once to the topics themselves.

1. The Simple Linear Production Model

At this point the reader should refresh his memory on the definition of a *linear production model.* He will recall that, if such a model involves n goods $G_1, \ldots, G_n$ and m activities $P_1, \ldots, P_m$, then the model is completely described by the production matrix $A = (\alpha_{ij})$, where α_{ij} is the amount of G_j consumed or produced by activity P_i according as α_{ij} is negative or positive.

The above is a description of a *linear production model.* A *simple linear production model* will naturally be a special case of the general model. The special assumptions are these:

Assumption I. *Each activity P_i produces only one good G_j.* In more familiar terms, we are assuming that there is no joint production, and there are no by-products of any activity. In terms of the matrix A the assumption means that there is only one positive entry in each row a_i, all the rest being zero or negative.

Assumption II. *Each good G_j is produced by one and only one activity P_i.*

This means, in particular, that there are the same number of activities as goods, and it is natural to label goods and activities correspondingly. We shall agree henceforth that P_i is the activity which produces G_i. The production matrix A for a simple model is square.

Because of Assumptions I and II it is convenient to modify slightly the definition of the matrix A as follows: Let us agree that α_{ij} *shall stand for the amount of G_j which it is necessary to consume in order to produce one unit of G_i.* Since consumption is now being taken as positive rather than negative it is appropriate to refer to A as the *consumption matrix* of the model. The ith row a_i of A gives the inputs of various goods required to produce one unit of G_i. We do not exclude the possibility that α_{ii} is positive; that is, it may be necessary to consume a certain amount of steel in order to produce more steel. Thus a *consumption* matrix A for a simple linear model may be any nonnegative square matrix.

We shall be concerned first of all with a simple feasibility question. Suppose the model with matrix A is asked to produce the "bill of goods" $y = (\eta_1, \ldots, \eta_n)$, that is, to produce η_1 units of G_1, η_2 units of G_2, etc. Does there exist a production program so that this demand can be met? Now if the activity P_i is operated at *level* or *intensity* ξ_i (we are using the terminology of Chap. 1) then ξ_i units of G_i will be produced. At the same time P_i will consume the vector $\xi_i a_i$ and the amounts consumed by the whole model will clearly be

$$\sum_{i=1}^{n} \xi_i a_i = xA$$

where $x = (\xi_i)$ gives the levels at which each activity is operated. Then *net production*, that is, production minus consumption, is given by the vector

$$x - xA = x(I - A)$$

and the feasibility question is simply: Given $y \geqq 0$ does the equation

$$x(I - A) = y \tag{1}$$

have a nonnegative solution?

It is clear that (1) need not have a solution. For example, let A be given by

$$A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

Then
$$(I - A) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

and certainly
$$x(I - A) \geq 0$$

has no nonnegative solution. We could have reached this conclusion by simple common sense. If it takes 2 units of G_1 to make 1 unit of G_2 and also 2 units of G_2 to make 1 unit of G_1 then one cannot produce positive amounts of both G_1 and G_2 simultaneously. We see then that if our technology is going to be useful at all it must be capable of producing at least one positive output vector. We are thus led to make the following

Definition. A simple linear model with consumption matrix A will be called *productive* if there exists a nonnegative vector $\bar{x}$ such that $\bar{x} > \bar{x}A$. We shall also say in this case that the matrix A itself is *productive.*

The key property of simple production models is the fact that if such a model is productive then it is capable of producing any positive output vector y, that is,

Theorem 9.1. *If the matrix A is productive then for any* $y \geqq 0$ *the equation*

$$x(I - A) = y$$

has a unique nonnegative solution.

The theorem will be a consequence of the following lemma:

Lemma 9.1. *If A is productive and* $x \geqq xA$ *then* $x \geqq 0$.

Proof. By definition there is a vector $\bar{x} = (\bar{\xi}_1, \ldots, \bar{\xi}_n) \geqq 0$ such that $\bar{x} > \bar{x}A$. This means that $\bar{\xi}_j > \bar{x}a^j$ and hence $\bar{\xi}_j > 0$; so $\bar{x} > 0$. Suppose now that $x = (\xi_1, \ldots, \xi_n)$ satisfies $x \geqq xA$ but $x \ngeqq 0$. Then some coordinate of x is negative. Let $\theta = \max [-\xi_i/\bar{\xi}_i]$, say $\theta = -\xi_1/\bar{\xi}_1$. Then θ is positive and $x' = x + \theta\bar{x} = (\xi_1', \ldots, \xi_n') \geqq 0$, with $\xi_1' = 0$. But also $x' = x + \theta\bar{x} > xA + \theta\bar{x}A = x'A \geqq 0$, which would imply $\xi_1' > x'a^1 \geqq 0$, a contradiction.

Corollary. If A is productive then $I - A$ is regular (has rank n).

Proof. If $x(I - A) = 0$ then $-x(I - A) = 0$, but by the lemma this means $x \geqq 0$ and $-x \geqq 0$ and therefore $x = 0$.

Proof of Theorem. Since $I - A$ is regular there exists a unique x such that $x(I - A) = y$ and since $y \geqq 0$ the lemma implies $x \geqq 0$.

Corollary. The matrix A is productive if and only if $(I - A)^{-1}$ is nonnegative.

Proof. The ith row of the matrix $(I - A)^{-1}$ is the vector x_i such that $x_i(I - A) = u_i$, and we have just seen that x_i must be nonnegative. Conversely, if $(I - A)^{-1}$ exists and is nonnegative then $x = u(I - A)^{-1}$ is nonnegative; so $x(I - A) = u > 0$ and A is productive (u is once again the unit vector).

This corollary gives a simple means for deciding whether A is productive. One simply computes the inverse of $I - A$.

We call attention to the fact that Theorem 9.1 makes strong use of the condition that there is no joint production, given by Assumption I. If, for example, we had a production matrix of the form

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

where positive numbers now represent outputs, then clearly it would not be possible to produce the vector (1, 0).

The reader may object to this example since here the first activity is giving us "something for nothing." A more realistic counter-example is given in the exercises (see Exercise 4).

One might think that for the sake of greater generality one should consider not only productive matrices but also *semiproductive* ones, i.e., matrices A such that $x \geq xA$ for some $x \geqq 0$. It turns out, however, that this case can immediately be reduced to the previous one, as we now show.

Definition. If A is semiproductive we call the activity P_j *productive* if there is a vector $x = (\xi_1, \ldots, \xi_n) \geqq 0$ such that $x \geqq xA$ and $\xi_j > \sum_{i=1}^{n} \alpha_{ij}\xi_i$. The corresponding good G_j is called *producible.* The remaining activities and goods will be called *nonproductive* and *nonproducible,* respectively.

Theorem 9.2. *If P_i is productive and P_j is nonproductive then* $\alpha_{ij} = 0$.

In the terminology of the previous chapter, the indices j corresponding to productive processes form an independent subset. In more economic language, the theorem says that no nonproducible good is used in any productive process. Therefore, by reordering we may make the production matrix take the form

$$\begin{matrix} \text{Productive} \\ \text{Nonproductive} \end{matrix} \begin{Bmatrix} \\ \end{Bmatrix} \overbrace{\left(\begin{array}{c|c} A_1 & 0 \\ \hline \multicolumn{2}{c}{A_2} \end{array} \right)}$$

where the matrix A_1 is square.

Proof. Since P_i is productive there exists $x = (\xi_1, \ldots, \xi_n) \geqq 0$ such that

$$\xi_i > \sum_{k=1}^{n} \xi_k \alpha_{ki} \qquad \text{in particular } \xi_i > 0 \tag{1}$$

and

$$\xi_r \geqq \sum_{k=1}^{n} \xi_k \alpha_{kr} \qquad \text{for all } r \tag{2}$$

Since P_j is nonproductive we must have

$$\xi_j = \sum_{k=1}^{n} \xi_k \alpha_{kj} \tag{3}$$

Now, for ϵ small and positive we may replace ξ_i by $\xi_i - \epsilon$ without disturbing inequality (1) while inequalities (2) are, if anything, strengthened; and if $\alpha_{ij} > 0$ then (3) becomes

$$\xi_j > \sum_{k=1}^{n} \xi_k \alpha_{kj} - \epsilon \alpha_{ij} \tag{4}$$

But this would mean P_j is productive contrary to hypothesis.

The economic moral of this theorem is that there is nothing to be gained by operating the nonproductive processes. Namely, let $x = (\xi_1, \ldots, \xi_n)$ be an intensity vector such that $x \geqq xA$, or $\xi_j \geqq \sum_{i=1}^{n} \xi_i \alpha_{ij}$.

If we set $\xi_i = 0$ for all nonproductive i then if P_j is productive we possibly strengthen the above inequalities, while if P_j is nonproductive we get $0 = 0$; so this new intensity vector is just as good as, and possibly better than, the original one.

So far our discussion has been concerned entirely with technology. We turn now to economic considerations by introducing prices. As usual we let $p = (\pi_1, \ldots, \pi_n)$ be the price vector, where π_j is the price of one unit of G_j. Then the *profit* of the activity P_i is given by the expression $\pi_i - \sum_{j=1}^{n} \alpha_{ij}\pi_j$, and the *profit vector* q is given by $q = (I - A)p$. We recall that in the linearexchange models of the previous chapter it was not possible for all activities to make a profit simultaneously. Here the situation is quite the opposite, as we now see.

Theorem 9.3. *If A is productive then for any nonnegative (profit) vector q there exists a unique nonnegative (price) vector p such that $q = (I - A)p$.*

Proof. Since $I - A$ is regular there exists a unique p such that $q = (I - A)p$, and it only remains to show that p is nonnegative. From Theorem 9.1 there exists $x_i \geqq 0$ such that $x_i(I - A) = u_i$; so $0 \leqq x_iq = x_i(I - A)p = u_ip$. Since all coordinates of p are nonnegative, the result follows.

The theorem shows that if prices are appropriately set the profits of the various activities may be any preassigned nonnegative numbers.

2. *A Dynamic Property of the Simple Model*

Let us suppose now that we are dealing with a simple model whose matrix is A, and suppose further that outside consumers have demanded a set of goods given by the nonnegative vector y_0. If we think of our model as being operated by a central planning authority then there will be no difficulty in meeting the demand y_0 assuming that A is productive. The planning authority simply solves the equation

$$x(I - A) = y_0$$

and if the solution is $\bar{x} = (\bar{\xi}_i)$ then each activity P_i is ordered to operate at the level $\bar{\xi}_i$.

However, we may also think of the activities P_i as being operated by completely independent authorities none of whom has any control over the actions of the others. In this case we can nevertheless describe a method by which the production problem can be solved at least approximately. The method is the following: Let us denote by $y_0 = (\eta_{i0})$ the initial demand by outside consumers. Then, in particular, there is a demand for η_{10} units of G_1. Now if this amount is to be produced by P_1 then P_1 must consume the vector $\eta_{10}a_1$. Thus having received an order for η_{10} units of G_1 the operator of P_1 in turn places orders for other goods in the amounts $\eta_{10}a_1$. Similarly, all the other producers P_i place orders for the vectors $\eta_{i0}a_i$. If we think of the vector y_0 as the first round of orders we see that in order to fill these orders the producers must initiate a second round of orders for the total vector

$$y_1 = \sum_{i=1}^{n} \eta_{i0}a_i = y_0A$$

But now, in order to fill the orders $y_1 = (\eta_{i1})$ the producers must place a third round of orders for the vector

$$y_2 = \Sigma\eta_{i1}a_i = y_1A = y_0A^2$$

and it is clear that this reordering process will go on indefinitely. The total bill of goods ordered will be the sum of those ordered at each stage, and we are thus led to consider the infinite series of vectors

$$y_0 + y_0A + \cdots + y_0A^n + \cdots$$

If we are fortunate, then this series will, in fact, converge and give us the correct amount which each P_i should produce in order for the model as a whole to supply the initial demand y_0.

Of course, in reality no such infinite sequence of reorderings could actually occur and the description above is not to be taken too literally. On the other hand, one can imagine some process like the one described taking place over a long period of time.

The theorem we shall prove is the following:

Theorem 9.4. *Let y be any vector and let $x_n = y(I + A + \cdots + A^n)$. Then if A is productive*

$$\lim_{n\to\infty} x_n(I - A) = y$$

The theorem shows that if we go through the reordering routine sufficiently many times we shall come arbitrarily close to satisfying the demand y. The proof is an easy consequence of the following

Lemma 9.2. *If A is productive then* $\lim_{n\to\infty} A^n = 0$.

Proof. Since A is productive there exists $x > 0$ such that $xA < x$. In fact there exists λ such that

$$xA < \lambda x \qquad (1)$$

where $0 < \lambda < 1$. From (1) it follows by induction that

$$xA^n < \lambda^n x$$

and hence $\lim_{n\to\infty} xA^n = 0$. But letting $x = (\xi_i)$ we have

$$xA^n = \Sigma\xi_i(u_iA^n) \to 0$$

Since all terms in the sum are nonnegative each must approach zero; hence (u_iA^n) approaches zero for all i and therefore A^n approaches zero as asserted.

Proof of Theorem. We simply observe that

$$\begin{aligned} x_n(I - A) &= y(I + A + \cdots + A^n)(I - A) \\ &= y - yA^{n+1} \end{aligned}$$

and since yA^{n+1} approaches zero, the theorem is proved.

3. *The Leontief Model*

One of the unnatural features of the simple model discussed in the preceding sections was the fact that if the model was capable of producing any positive goods vector then it could produce arbitrarily large amounts of any of the goods in any proportions. There is nothing wrong with this provided the model is given a sufficient amount of time in which to do the producing. In realistic production problems, however, time is generally of the essence. When consumers demand some goods vector y they expect to receive the goods not in the indefinite future but within some specified time, say a year.

In order to go over to a more realistic model which takes account of time, we need change nothing in our mathematical model but merely our interpretation of the quantities involved. The consumption number α_{ij} now becomes the amount of G_j required, say *per year*, in order to obtain a *yearly* output of one unit of G_i. In this interpretation it is obviously unreasonable to expect to be able to produce arbitrarily large quantities in a limited amount of time. Why? Because of limitations on *capacity*. No matter how much is available in the way of steel, say, for making automobiles, there are only enough machines, equipment, and especially *labor* to produce a certain finite number of cars per year.

The above ideas are easily formalized. Goods like plant equipment and labor are characterized by the following two properties:

1. They are not outputs of any of the activities of the model.
2. They are available in a limited amount.

Goods satisfying (1) and (2) are called *primary goods* (also sometimes referred to as *factors of production*).

We are now prepared to describe the *simple Leontief model.*

Definition. The *simple Leontief model* consists of a simple production model in which there is a single primary good G_0 called *labor.*

We shall assume that labor is needed as an input to all activities; that is, the consumption coefficients α_{i0} are all positive. We also choose the unit of labor so that the total amount available is 1.

The first question for the Leontief model which can be quickly settled is that of prices. For the case of the simple model without primary goods we have seen that profits could be any numbers at all. For the Leontief model with labor as primary input it is natural to assume that when the cost of labor is taken into account the profit of each activity shall be zero. In other terms, all the profit which is made from production is turned back to labor as wages.

Theorem 9.5. *There exists a positive price vector p, unique up to multiplication by a positive number, such that at prices p the profit to each activity is zero.*

Proof. Let us assume the price of labor π_0 to be 1. The condition that profits be zero is then

$$\alpha_{i0} + \sum_j \pi_j \alpha_{ij} = \pi_i \qquad \text{for all } i > 0$$

or

$$(I - A)p = a^0 \tag{1}$$

where $a^0 = (\alpha_{i0})$ and A is the consumption matrix without the column a^0. By Theorem 9.3, (1) has a unique nonnegative solution, and since a^0 is positive p must also be positive.

So far we have not made any explicit use of Assumption II. This was the condition that there be *only one* process for making each commodity. In realistic situations there might be a number of alternative ways of producing a given good. Let us define a *general* (as opposed to simple) *Leontief model* to satisfy all the conditions imposed on the simple model except that the good G_j may be producible by more than one activity. In a general model, then, there will be more activities than goods. We denote by S_j the set of all activities which produce G_j or, more conveniently, the set of all indices i such that P_i produces G_j. Then, given an intensity vector $x = (\xi_i)$ determining the various activity levels and such that the labor supply is not exceeded, the corresponding net *output vector* $y = (\eta_j)$ is given by

$$\eta_j = \sum_{i \varepsilon S_j} \xi_i - \sum_{i=1}^{m} \xi_i \alpha_{ij} \qquad \text{where } \sum_{i=1}^{m} \xi_i \alpha_{i0} \leqq 1$$

The set Y of all such vectors will be termed the *output space* of the model.

It might reasonably be expected that the analysis of the general Leontief model would be considerably more involved than that of the simple model. It turns out somewhat surprisingly, however, that almost all questions concerning general models can be reduced to questions about certain simple submodels, as the following interesting theorem shows.

Theorem 9.6 (substitution theorem). *If a general Leontief model is productive then there exists a set of n activities $P_{i_1}, \ldots, P_{i_n}$, where $i_j \varepsilon S_j$, such that the simple Leontief model formed from these activities has the same output space as the original model.*

There are a number of ways of proving this result. The most instructive method is via the duality theory of linear programming.

We digress for a moment, therefore, to reconsider the canonical minimum problem of finding a nonnegative vector x which

$$\text{minimizes } xc \tag{1}$$

subject to

$$xA = b \tag{2}$$

We call a set of independent rows a_i of A an *optimal* (*feasible*) *basis* if there is an optimal (feasible) vector x depending on these rows. The result we need is the following:

Lemma 9.3. *Let a set* $\mathcal{B}$ *of rows* a_i *of* A *be an optimal basis for the problem* (1), (2) *above and consider the new problem*

$$\textit{to minimize } xc \tag{1'}$$

subject to

$$xA = b' \tag{2'}$$

Then if $\mathcal{B}$ *is a feasible basis for problem* (1′), (2′) *it is, in fact, an optimal basis for this problem also.*

Proof. Let $\bar{x}$ be an optimal vector for problem (1), (2) depending on the basis $\mathcal{B}$ and let $\bar{y}$ be a solution of the dual. Then we know from the Equilibrium Theorem 3.2 (page 82) that

$$\text{if } a_i\bar{y} < \gamma_i \qquad \text{then } \bar{\xi}_i = 0 \tag{3}$$

Now assume x' is a feasible vector for problem (1′), (2′) depending on the set $\mathcal{B}$. Then we have also

$$\text{if } a_i\bar{y} < \gamma_i \qquad \text{then } \xi_i' = 0 \tag{3'}$$

since by hypothesis $\xi_i' = 0$ whenever $\bar{\xi}_i = 0$. But this is precisely the condition that x' and $\bar{y}$ be solutions of the primal and dual problems of (1′) and (2′) (again by Theorem 3.2) and, in particular, x' is an optimal vector, as asserted.

Proof of Substitution Theorem. Let $\bar{y}$ be a positive vector in the output space Y. We consider the canonical minimum problem of producing the vector $\bar{y}$ while minimizing the amount of labor used, that is,

$$xa^0 = \text{minimum}$$

Now let $\bar{x}$ be a basic optimal vector for this problem. Then $\bar{x}$ depends on at most n rows $i_1, \ldots, i_n$ of the production matrix. Since $\bar{y}$ is positive all goods are produced by $\bar{x}$ and hence ξ_i must be positive for one index i in each of the sets S_j. Letting $\bar{A}$ be the matrix with rows a_{i_j} it remains to show that $\bar{A}$ has the output space Y. Let y' be any other vector in Y. We know that the matrix $\bar{A}$ is productive since it produced the positive vector $\bar{y}$. Hence, by Theorem 9.1, there exists a vector x' such that

$$x'(I - \bar{A}) = y'$$

but this simply says that the basis given by the rows a_{i_j} is feasible for the new linear program where the vector to be produced is y' rather than $\bar{y}$. According to the previous lemma, therefore, x' is also optimal, that is,

$$x' \text{ minimizes } xa^0$$

among all possible vectors x for the original model. Since $y' \varepsilon Y$ there is some intensity vector x which produces y' and for which $xa^0 \leqq 1$ (the labor supply is not exceeded). It now follows that

$$x'a^0 \leqq 1$$

and hence y' is producible with the matrix $\bar{A}$ and one unit of labor. The proof is now complete.

There are a number of remarks to be made concerning the above result.

Remark 1. Notice that we have actually proved more than is stated in the theorem. Not only are we able to produce everything with the simple model that we could with the general model but we can also produce just as economically with the simple model in the sense that the amount of labor required for each output vector is no greater for the simple model than for the general. Thus, in a "Leontief economy," there is nothing to be gained by having several processes for producing the same good.

Remark 2. Both the primal and dual linear programming problem used in the proof of the substitution theorem have very natural economic interpretations. The primal problem asks for a way of producing a given bill of goods with minimum labor. The dual asks for a set of prices which will maximize the cost of this bill of goods subject to the familiar condition that no activity shall show a positive profit.

Remark 3. The validity of the substitution theorem depends strongly on the assumption that there is only one primary good, as the following counter-example shows: We consider a model in which there are two primary goods, say *skilled labor* G_0 and *unskilled labor* G'_0. There are also two output goods G_1 and G_2. There is only one activity P_1 producing G_1, and this requires an input of one unit of skilled labor. There are two activities P_2 and P'_2 for producing G_2. The first requires an input of 1 unit of skilled labor; the second requires an input of 2 units of unskilled labor. We leave it to the reader to show that in this example if either of the activities P_2 or P'_2 are eliminated the output space becomes smaller (see Exercise 10).

Finally, we observe that the way to find the simple submodel with the properties of the substitution theorem is to solve the linear program for minimizing labor input. The reader will do well at this point to work the numerical example of Exercise 9.

4. *The General Linear Production Model. Efficient Points*

We turn now to the consideration of the most general linear production model in which we place no restrictions on the nature of the production activities. An activity may have any number of outputs as well as inputs, the same good may be produced by any number of activities, and there is no restriction on the allowable number of primary goods. In formulating the model it is actually more convenient not to distinguish between primary and final goods but to use an alternative description.

We consider the usual linear activity model with *production matrix* which for reasons which will soon be apparent we shall denote by B. The m rows and n columns correspond to the usual activities and goods, respectively, and for a given nonnegative *input vector* x we obtain an *output vector* y where

$$y = xB \tag{1}$$

Now, because of the existence of primary goods not all nonnegative vectors x may be possible, as we have seen, for example, in the case of the Leonteif model where we have to satisfy the condition

$$xa^0 \leqq 1$$

The generalization to more than one primary good is obvious. We are given a *consumption matrix* A and a *supply vector* b and we require that

$$xA \leqq b \tag{2}$$

The matrix A of course has m rows. The columns, however, will correspond only to those goods of which there is a limited supply.

The model described here is entirely similar to the one treated in Sec. 5, Chap. 3. The inequality (2) corresponds to what we called limitations of plant capacity in our earlier discussion. We have now given a complete description of the model and we turn to a study of some of its properties.

Definition. The nonnegative solutions of (2) above will be called the *input space* of the model, denoted by X.

The *output space* Y of the model consists of all vectors y such that $y = xB$ for x in X (we do not require that y be nonnegative).

We now recall briefly some terminology from Chap. 2. The input space above, being the set of all solutions of a set of inequalities, was called a *solution set* (see Exercise 36, Chap. 2). It follows from Exercise 40 of Chap. 2 that the output space Y is also a solution set. This means that there exist some n-rowed matrix C and some vector c such that Y is the set of all solutions of the inequalities

$$yC \leqq c \tag{3}$$

It is this characterization of the output space Y which we shall make use of in what follows.

We come now to the central economic notion of this section.

Definition. The vector y_0 in Y is called *efficient* if there is no vector y' in Y such that $y' \geq y$.

In words, y_0 is efficient if it is impossible to increase the output of any good without decreasing the output of some other.

The principal result on efficient points relates them to prices and income. It seems intuitively obvious on economic grounds that any output vector which maximizes income at some set of prices will be efficient. Conversely, it turns out that corresponding to each efficient vector y_0 there is a set of prices for which y_0 is an income maximizer, as we now show.

Theorem 9.7. *If Y is the output space of a general linear model then the vector y_0 in Y is efficient if and only if there exists a positive (price) vector p such that $y_0p \geqq yp$ for all $y \in Y$.*

Proof. If p is given and $y_0p \geqq yp$ for all y in Y then clearly y_0 is efficient, for if not we would have $y' \in Y$ and $y' \geq y_0$ and therefore $y'p > y_0p$, contrary to assumption.

Conversely, suppose y_0 is efficient. From (3) above we have

$$y_0c^j \leqq \gamma_j \qquad \text{for all } j$$

We now divide the indices j into two sets S and S' where

$$\begin{aligned} y_0c^j &= \gamma_j \qquad \text{for } j \in S \\ y_0c^j &< \gamma_j \qquad \text{for } j \in S' \end{aligned}$$

The set S cannot be empty, for then we would have $y_0C < c$, so that for a sufficiently small positive number ϵ we would have

$$(y_0 + \epsilon v)C \leqq c$$

and $y_0 + \epsilon v$ would be in Y, contradicting the assumption that y_0 is efficient.

We next assert that the inequalities

$$zc^j \leqq 0 \qquad \text{for all } j \in S \tag{4}$$

have no semipositive solution, for if z were such a solution then clearly we would have, for any positive number ϵ,

$$(y_0 + \epsilon z)c^j \leqq \gamma_j \qquad \text{for } j \in S$$

and for ϵ sufficiently small

$$(y_0 + \epsilon z)c^j \leqq \gamma_j \qquad \text{for } j \in S'$$

but this would mean that $y_0 + \epsilon z$ is in Y, again contradicting the efficiency of y_0.

Now since inequalities (4) have no semipositive solution we know from Theorem 2.10 (page 49) that there are numbers λ_j, $j \varepsilon S$ such that

$$\sum_{j \varepsilon S} \lambda_j c^j > 0$$

Letting $p = \sum_{j \varepsilon S} \lambda_j c^j$, the proof is completed by noting that, for y in Y,

$$yp = \sum_S \lambda_j (yc^j) \leqq \sum_S \lambda_j \gamma_j = \sum_S \lambda_j (y_0 c^j) = y_0 p$$

so that y_0 maximizes income at prices p.

The above theorem gives a mathematical justification for some of the tenets of classical price theory. It states that any efficient mode

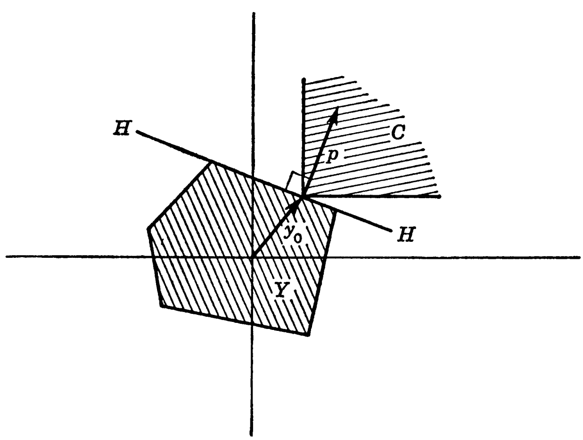

FIG. 9.1

of production can be achieved by setting prices appropriately and allowing producers to maximize their income.

There is a simple geometric picture corresponding to this theorem. Let y_0 be efficient in Y. The set of all vectors y' such that $y' \geq y_0$ clearly form a convex cone C with vertex at y_0. The efficiency of y_0 states that C and Y do not intersect (see Fig. 9.1). It is geometrically obvious then that there is a hyperplane H through y_0 which separates

C from Y, and the normal to this hyperplane in the direction of C gives the desired price vector.

Finally we remark that neither the efficient point corresponding to given prices nor the prices for a given efficient point need be unique (see Exercises 11, 12, 13).

5. von Neumann's Expanding Model

The previous sections have been concerned with the static theory of production models. We now take up the case of a model whose output varies with time and consider the possibility of a steady expansion of such a model in relation to prices of the goods involved.

As in the previous section we consider a general linear model involving n goods $G_1, \ldots, G_n$ and m activities $P_1, \ldots, P_m$. We wish now to distinguish explicitly between goods produced and consumed in a given activity. Accordingly we denote by α_{ij} the amount of G_j consumed in P_i, and by β_{ij} the amount of G_j produced by P_i. The activity P_i is accordingly represented by two nonnegative vectors, an *input vector* $a_i = (\alpha_{i1}, \ldots, \alpha_{in})$ and an *output vector* $b_i = (\beta_{i1}, \ldots, \beta_{in})$. The corresponding matrices $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ are called the *input* and *output* matrices, respectively. Once again, an *intensity vector* x is a semipositive m-vector, and the corresponding input and output vectors are given by xA and xB. Henceforth we shall denote the model symbolically by the pair (A, B).

The model we are now considering is assumed to be *closed.* This means that there is no flow of goods to or from the model. All goods consumed in the model must have been previously produced by it, and the only thing one can do with the model's output is to feed it back into the model as an input at a later stage. We are therefore concerned with a sort of self-sustaining mechanism whose sole function is to perpetuate itself in some manner. Such a model is in some sense an approximation to a total economy in which labor produces consumption goods and these goods are then consumed by consumers, enabling or inducing them to give more labor, so that we have roughly the cyclic situation described by the model.

It is clear that in order for the model to function in the manner described it must be possible to choose an intensity vector x so that every good which is consumed is also produced. A condition sufficient to ensure this is

Assumption I. For all j, $b^j \geq 0$.

This simply states that every good is an output of some activity.

We shall also make a second assumption which asserts that it is impossible to get something for nothing.

Assumption II. For all i, $a_i \geq 0$.

This says that every activity must have at least one good as input.

Now suppose we have an intensity vector x such that $xB \geqq \alpha xA$ for some number $\alpha > 0$. We then say that the *model is expanding at a rate at least equal to* α. The terminology is appropriate, for the above inequality means that $xb^j \geqq \alpha xa^j$; thus the output of each good G_j is at least α times as great as its input. Note that this does not exclude the possibility that both input and output of some good be zero.

Definition. *The technological expansion problem* for the model (A, B) is to find a semipositive m-vector x and number α such that

$$\alpha \text{ is a maximum} \tag{1}$$

subject to

$$xB \geqq \alpha xA \tag{2}$$

If the maximum value of α exists it is called the *technological expansion rate*[1] of the model and is denoted by α_0. The corresponding intensity vector x_0 is then called an *optimal intensity vector.*

Theorem 9.8 (existence theorem). *For models satisfying Assumptions* I *and* II, α_0 *exists and is positive.*

Proof. For each positive number α, consider the problem of finding a semipositive solution x to the inequality

$$x(B - \alpha A) \geqq 0 \tag{3}$$

Now, for α sufficiently small (3) has a solution, for if x is positive

[1] Although we use the term "expansion" throughout, we nowhere require that α be greater than unity. Thus the whole theory of these models applies equally well to "contracting" models, although the case $\alpha \geqq 1$ is the one of economic interest.

it follows from Assumption I that xB is positive; hence $xB \geqq \alpha xA$ for some positive number α. On the other hand, for α very large (3) has no solutions, for since there is a positive entry in every row of A (Assumption II), we can choose α so large that the sum of the coordinates in each row of $B - \alpha A$ is negative, that is,

$$(B - \alpha A)v < 0$$

where v is the unit vector in R^n, and hence for any $x \geq 0$,

$$x(B - \alpha A)v < 0$$

so that (3) has no solution.

Let α_0 be the least upper bound of all numbers for which (3) has a solution; it is clear that α_0 is the desired technological expansion rate.[1]

Having considered so far only the technological aspects of the model, we turn now to the economic theory. We shall obtain results here which are striking analogues to the duality theorems of linear programming.

As usual, we consider a semipositive price vector $p = (\pi_1, \ldots, \pi_n)$. At prices p the *cost* of activity P_i is a_ip and the *revenue* from P_i is b_ip. The *cost vector* is Ap; the *revenue vector* is Bp. We now describe an economic problem which will turn out to be the dual of the expansion problem previously defined.

Definition. The *economic expansion problem* for the model (A, B) is to find a semipositive n-vector p and number β such that

$$\beta \text{ is a minimum} \qquad (1)^*$$

subject to

$$Bp \leqq \beta Ap \qquad (2)^*$$

The minimum value of β is called the *economic expansion rate* of the model and is denoted by β_0. The corresponding price vector p_0 is then called an *optimal price vector.* We remark that from Assumption I, for any price vector p, $b_ip > 0$ for some index i and therefore

[1] A standard "compactness" argument is needed here to show that there actually exists a semipositive x_0 such that $x_0(B - \alpha_0 A) \geqq 0$.

$\beta_0 > 0$. An argument analogous to that of Theorem 9.8 shows that β_0 always exists.

There are various possible economic interpretations of the dual problem. If $a_i p > 0$ then the ratio $b_i p / a_i p$ is return divided by cost, which is the rate at which the value of goods is increasing in activity P_i, a sort of profit factor. Now if one makes the assumption of a free competitive economy, the forces of competition between activities will tend to make this profit factor a minimum. A second interpretation of β is as an *interest factor.* Suppose the activities are financed by borrowing and that at the end of each period of production, for each dollar borrowed, the activities must pay back β dollars (the usual interest rate would be $\beta - 1$). Then condition (2)* is the familiar condition that no activity shall make a profit.

We remark that, although the pair of dual problems treated here seem very similar to dual linear programming problems, there is a fundamental difference in that the constraints are not linear. In fact we shall see that a model all of whose matrix coefficients are integers may have expansion rates which are irrational.

We also make the trivial observation that both our problems are homogeneous, so that if x_0 or p_0 is optimal so also is any positive multiple of either.

We now proceed to derive the relationship between the dual problems. The situation is somewhat more complicated than the case of linear programming.

Lemma 9.4. *For models satisfying Assumptions* I *and* II, $\beta_0 \leqq \alpha_0$. (We shall see from examples that equality need not hold.)

Proof. Define the matrix C by $C = B - \alpha_0 A$. Now the inequality $xC > 0$ has no nonnegative solution, for if x' were such a solution then we would have $x'B > \alpha_0 x'A$, or $x'B \geqq (\alpha_0 + \epsilon)x'A$ for some $\epsilon > 0$ so that α_0 would not be maximal. Now apply Theorem 2.10 (page 49), which asserts that there exists $p \geq 0$ such that $Cp \leqq 0$; so $Bp \leqq \alpha_0 Ap$. Therefore, from the definition it follows that $\alpha_0 \geqq \beta_0$, as was to be shown.

Although one cannot assert that $\alpha_0 = \beta_0$ without some further assumptions, there is an interesting analogue of the linear programming Equilibrium Theorem 1.2, which we now give.

Theorem 9.9 (von Neumann). *If the model* (A, B) *satisfies* I *and* II *then there exists a semipositive m-vector* $x_0 = (\xi_1, \ldots, \xi_m)$, *a semipositive n-vector* $p_0 = (\pi_1, \ldots, \pi_n)$, *and a number* γ *such that*

$$x_0B \geqq \gamma x_0A \qquad \text{(i)}$$

and $$\text{if } x_0b^j > \gamma x_0a^j \quad \text{then } \pi_j = 0 \qquad \text{(ii)}$$

$$Bp_0 \leqq \gamma Ap_0 \qquad \text{(i)*}$$

and $$\text{if } b_ip_0 < \gamma a_ip_0 \quad \text{then } \xi_i = 0 \qquad \text{(ii)*}$$

Proof. Let $\gamma = \alpha_0$, the technological expansion coefficient, and let x_0 and p_0 be optimal intensity and price vectors. Then (i) holds by definition of α_0. Also $Bp_0 \leqq \beta_0Ap_0 \leqq \alpha_0Ap_0 = \gamma Ap_0$, using Lemma 9.4; so (i)* is established.

Next, from (i) and (i)* we have $\gamma x_0Ap_0 \leqq x_0Bp_0 \leqq \gamma x_0Ap_0$; so $x_0(B - \gamma A)p_0 = 0$, or $\Sigma\xi_i(b_i - \gamma a_i)p_0 = 0 = \Sigma x_0(b^j - \gamma a^j)\pi_j$, and since the terms $(b_i - \gamma a_i)p_0$ and $x_0(b^j - \gamma a^j)$ are all nonnegative we obtain (ii) and (ii)*.

Interpretation. The constant γ is both expansion rate and interest factor. Condition (i) states that the amounts of all goods increase at a rate at least equal to γ. Condition (i)* is the requirement that no activity show a positive profit. Condition (ii) states that if any good is expanding at a rate greater than γ then, being "oversupplied," its price drops to zero, and (ii)* is the obvious condition that activities which show a negative profit will not be used.

In order to obtain the full duality theorem we need a further concept which is a generalization of the notion of independent subset of the preceding chapter.

Definition. Given a model (A, B) the set of indices $S \subset \{1, \ldots, n\}$ is called an *independent subset* if it is possible to produce each good G_i, $i \in S$, without consuming any good G_j, j not in S. More formally, the set S is independent if there exists a set $T \subset \{1, \ldots, m\}$ such that $\alpha_{ij} = 0$ for $i \in T$ and $j \in S'$ and for all $j \in S$, $\beta_{ij} > 0$ for some $i \in T$. The model is *irreducible* if there are no proper independent subsets.

If we reorder rows and columns of the matrix A so that the indices T correspond to the first t rows of A and the indices S to the first s columns then the matrix takes the form

$$t\left\{\left(\begin{array}{cc}\overbrace{A_1}^{s} & 0\\ \hline \multicolumn{2}{c}{A'}\end{array}\right)\right.$$

In economic terms, a set of goods S is independent if these goods can be produced from themselves, that is, without consuming any other goods.

Theorem 9.10 (duality theorem). *If the model* (A, B) *is irreducible then* $\alpha_0 = \beta_0$.

Proof. We have already shown that $\beta_0 \leqq \alpha_0$; so we need only show the reverse inequality. If x_0 and p_0 are optimal then $x_0B \geqq \alpha_0x_0A$ and $Bp_0 \leqq \beta_0Ap_0$ so $\alpha_0x_0Ap_0 \leqq x_0Bp_0 \leqq \beta_0x_0Ap_0$, and if we can show that $x_0Ap_0 > 0$ the desired inequality will follow. Letting S be all indices j such that $x_0b^j > 0$, we see that S is an independent subset, for taking T to be all indices i such that $\xi_i > 0$, we must have $\alpha_{ij} = 0$ for $i \in T, j \in S'$, for otherwise we would get $x_0a^j > 0$, $x_0b^j = 0$ and we could not have $x_0b^j \geqq \alpha_0xa^j$. From irreducibility, then, $x_0b^j > 0$ for all j, or $x_0B > 0$. Since $p_0 \geq 0$ it follows that $x_0Bp_0 > 0$ and therefore $\beta_0x_0Ap_0 \geqq x_0Bp_0 > 0$, which shows that x_0Ap_0 is positive, completing the proof.

6. *Some Examples*

Consider the model whose input and output matrices are the following:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}$$

This model is irreducible, as one easily verifies. We assert that the optimal intensity and price vectors are

$$\begin{aligned} x_0 &= (2^{-1/3}, 2^{-2/3}, 1)\\ p_0 &= (1, 2^{-1/3}, 2^{-2/3}, 0)\end{aligned}$$

for then the input and output vectors are

$$x_0A = (2^{-2/3}, 2^{-1/3}, 1, 2^{-2/3}) \qquad x_0B = (2^{-1/3}, 1, 2^{1/3}, 1)$$

and the corresponding expansion rate α is given by

$$\alpha = \min\, [2^{1/3}, 2^{1/3}, 2^{1/3}, 2^{2/3}] = 2^{1/3}$$

Also

$$Ap_0 = (2^{-1/3}, 1, 2^{-2/3}) \qquad Bp_0 = (1, 2^{1/3}, 2^{-1/3})$$

and the expansion rate β is

$$\beta = \max\, [2^{1/3}, 2^{1/3}, 2^{1/3}] = 2^{1/3}$$

Since $\alpha = \beta$ it follows from the duality theorem that this common value is α_0 and β_0. Note that the good G_4 is overproduced since its expansion rate is $2^{2/3}$, and consequently its price is zero in accordance with the theory.

One can easily construct models for which $\beta_0 < \alpha_0$. If one simply "puts together" two models which have no activities or goods in common and which have different expansion rates then for the composite model α_0 will be the larger, β_0 the smaller of the expansion rates. A somewhat more complicated example of nonuniqueness is the following:

$$\overset{A}{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}} \quad \overset{B}{\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}}$$

Note that the "submodel" consisting of the last three rows is exactly the model of the previous example and therefore has expansion coefficient $2^{1/3}$. On the other hand, the submodel consisting of the first three rows has expansion coefficient 1, as can be seen by setting $x = (1, 1, 1, 0, 0)$, $p = (1, 1, 0, 0, 0, 0)$. Therefore $\alpha_0 = 2^{1/3}$, $\beta_0 = 1$.

7. *The Expanding Simple Model*

The simple linear model is a special case of the general model (A, B) in which there are exactly n activities and B is the identity matrix I. For simple models the expansion problem takes the form: Given a nonnegative matrix A, find a semipositive vector x and number α such that

$$\alpha \text{ is a maximum} \tag{1}$$

subject to

$$x \geqq \alpha x A \tag{2}$$

The simple model has the following important special property.

Theorem 9.11. *If x_0 and α_0 solve* (1) *and* (2) *above, then*

$$x_0 = \alpha_0 x_0 A$$

The theorem says that for a simple model expanding at a maximum rate there is no overproduction, all goods expanding at the maximum rate α_0.

Remark. We may rewrite the above equation in the form $x_0 A = (1/\alpha_0)x_0$. This shows that $1/\alpha_0$ is a positive *eigen-value* of A and x_0 is a corresponding nonnegative *eigen-vector.* Thus we are proving the classical result that a nonnegative matrix always has a nonnegative eigen-value and eigen-vector.

Proof. Suppose there were an optimal vector x_0 such that $x_0 \geq \alpha_0 x_0 A$. We cannot have $x_0 > \alpha_0 x_0 A$, for then α_0 could be replaced by a larger number. Now choose an optimal vector $x = (\xi_1, \ldots, \xi_n)$ such that the strict inequality

$$\xi_j > \alpha_0 x a^j = \alpha_0 \sum_{i=1}^{n} \xi_i \alpha_{ij} \tag{3}$$

holds for as many indices as possible, say all indices in the set S, while for the remaining indices S' we have

$$\xi_j = \alpha_0 x a^j = \alpha_0 \sum_{i=1}^{n} \xi_i \alpha_{ij} \qquad \text{for } j \in S' \tag{4}$$

We now assert

$$\xi_i \alpha_{ij} = 0 \qquad \text{for } i \in S, j \in S' \tag{5}$$

for if this were not the case then, say, $\xi_{i_0}\alpha_{i_0j_0} > 0$, $i_0 \in S$, $j_0 \in S'$. Then we could replace ξ_{i_0} by $\xi_{i_0} - \epsilon$, where ϵ is positive but sufficiently small so that inequalities (3) remain valid. But from (4)

$$\xi_{j_0} = \alpha_0 \sum_{i=1}^{n} \xi_i \alpha_{ij_0} > \alpha_0[\xi_1\alpha_{1j_0} + \cdots + (\xi_{i_0} - \epsilon)\alpha_{i_0j_0} + \cdots + \xi_n\alpha_{nj_0}]$$

so we would have a vector which increases the number of strict inequalities (3) contrary to the choice of x, and this proves (5).

Now let $x' = (\xi_1', \ldots, \xi_n')$, where $\xi_i' = \xi_i$ for $i \in S$, $\xi_i' = 0$ for $i \in S'$. Replacing x by x' in (3) we see that the inequalities are if anything strengthened, and in (4) we get $0 = 0$ because of (5). But this means that we could again increase α_0 contrary to its definition. Accordingly there are no strict inequalities (3) and the theorem is proved.

Using the above result one can now give a complete analysis of the possible optimal intensity and price vectors for the simple model. We state the results here. The proofs are precisely like those of Theorems 8.2 and 8.3 and are left as exercises.

Theorem 9.12. *If the matrix A is irreducible then $\alpha_0 = \beta_0$ and the optimal intensity vector is positive and unique up to multiplication by a positive number.*

If A is reducible with irreducible subsets $S_1, \ldots, S_k$, let A_i be the submatrix corresponding to S_i and let α_i and β_i be the technological and economic expansion coefficients, respectively, of A_i.

Theorem 9.13. *If A is as above then $\alpha_0 = \max [\alpha_i]$ and $\beta_0 =$* $\min [\beta_i]$.

Bibliographical Notes

The theorems on the simple linear production model are classical results in the theory of positive matrices. As presented here our proofs are similar to some of those given by Arrow [1]. The substitution theorem was first discovered by Samuelson [1], and a general proof was given by Arrow [1]. The one presented here based on duality was communicated to the author verbally by Dantzig. The relationship between efficient production and profit maximization is extensively developed by Koopmans in his fundamental paper [1]. The linear expanding model was introduced by von Neumann in a paper [3] which has been translated to English [4]. The particular formulation and results on the expanding model presented here were given by the author [3]. A somewhat different analysis has been given by Kemeny, Morgenstern, and Thompson [1].

Exercises

1. Show that the production matrix

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

is productive if and only if the *determinant*

$$|I - A| = (1 - \alpha_{11})(1 - \alpha_{22}) - \alpha_{12}\alpha_{21}$$

is positive.

2. Prove: If the matrix A is productive then the sum of the entries in at least one column is less than one.

3. Is the following matrix productive?

$$A = \begin{pmatrix} 0 & 0.4 & 1 \\ 1.2 & 0 & 0.3 \\ 0.2 & 0.2 & 0 \end{pmatrix}$$

4. Consider a 3×3 model with activity vectors

$$\begin{aligned} P_1 &= (4, -3, -1) \\ P_2 &= (-1, 3, -1) \\ P_3 &= (-2, 2, 3) \end{aligned}$$

Show that this model can produce some but not all positive output vectors. Why does this not contradict Theorem 9.1?

5. A matrix M is called *positive definite* if $xMx > 0$ for all $x \neq 0$ (see Chap. 2, Exercise 25). Show that if the production matrix $I - A$ of a simple model is positive definite then A is productive.

6. Show that if the consumption matrix A of a simple model satisfies

$$\lim_{n \to \infty} A^n = 0$$

then A is productive. [Hint: Show that $(I - A)^{-1}$ is nonnegative.]

7. Let the consumption matrix of a simple model be

$$A = \begin{pmatrix} 0.2 & 0.5 \\ 0.7 & 0.1 \end{pmatrix}$$

Find the input vector needed to produce one unit of each good. Compute x_n of Theorem 9.4 for $n = 3, 4, 5$ and compare with your answer above.

8. Show by an example that the substitution Theorem 9.6 is not valid for models in which there is joint production.

9. In a general Leontief model let the consumption matrix be the following:

	G_0	G_1	G_2
P_1	0.4	0.1	0.6
P_2	0.3	0.2	1.0
P_3	0.6	0.4	0
P_4	0.5	0.3	0.2

where P_1 and P_2 produce one unit of G_1
P_3 and P_4 produce one unit of G_2
G_0 is labor

Find a pair of these activities having the same output space as the original model.

What is the minimum amount of labor needed to produce one unit of each good?

10. Show that the substitution theorem does not hold in the example given in Remark 3 of Sec. 3.

11. Give an example of a linear production model involving 2 goods in which the point (1, 1) is efficient but there are infinitely many price vectors p for which (1, 1) maximizes income.

12. Give an example of a linear production model involving 2 goods in which there are many output vectors which maximize income at prices $p = (1, 1)$.

13. Let y be a vector in the output space Y of a linear model which maximizes income at prices p. Show that either y or p is *not* unique; that is, either there exists a vector $y' \neq y$ in Y which also maximizes income at prices p, or there is a price vector $p' \neq p$ such that y also maximizes income at prices p'. (Hint: Consider the finite cone generated by $Y - y$ and examine the dual cone.)

14. Consider the linear model whose input and output matrices are

the following:

A				B			
0	1	0	0	2	0	0	0
1	0	0	0	0	0	2	0
0	0	1	0	0	2	0	0
0	0	0	1	0	0	1	0
0	1	0	0	0	0	0	1

Is this model reducible? Find the expansion rates α_0 and β_0.

15. Show from the above example that if S_1 and S_2 are independent subsets (see definition in Sec. 5) then $S_1 \cap S_2$ need not be independent. What about $S_1 \cup S_2$?

16. Prove Theorems 9.12 and 9.13.

17. Show that for the simple expanding model if A is irreducible then the optimal price vector is positive and unique up to multiplication by a positive number.

18. Show that the expansion rate of an irreducible simple model is greater than 1 if and only if A is productive.

19. Find expansion rates and optimal intensity and price vectors for the simple model whose consumption matrix is

$$A = \begin{pmatrix} 0.3 & 0.5 \\ 0.6 & 0.4 \end{pmatrix}$$

20. Show that Theorem 9.3 is not true if either Assumption I or II does not hold.

21. In an expanding linear model involving n goods show that it is always possible to find an optimal intensity vector which depends on at most n activities. (Hint: Let $\xi_i > 0$ for $i \varepsilon S$. Then show that the vectors $b_i - \alpha a_i$, $i \varepsilon S$, belong to a subspace of dimension at most $n - 1$. Now use the theorem on basic solutions of equations.)

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