Problem Set on Convex Sets

This problem set is a collection of most of the facts about convex sets and convex and concave functions we will need. Some of these results are for you to prove — thus, a problem set.

Let's recall some definitions. These are all facts you should be familiar with. If not, or to brush up, these lecture notes by the late Kim Border are thorough and fun.

Half-Spaces

Definition 1. ***Closed half-spaces*** *in* $\mathbf{R^n}$ *are sets of points* $[p \geq \alpha] = \{x : p \cdot x \geq \alpha\}$ *and* $[p \leq \alpha] = \{x : p \cdot x \leq \alpha\}$ *for some* $p \in \mathbf{R^n}$ *and* $\alpha \in \mathbf{R}$*.* ***Open half-spaces*** *are sets of points* $[p \gg \alpha]\{x : p \cdot x \gg \alpha\}$ *or* $[p \ll \alpha]\{x : p \cdot x \ll \alpha\}$

A **half-space** is just the set of points one side or the other of a **hyperplane** $[p = \alpha] = \{x : p \cdot x = \alpha\}$. If the half-space includes the hyperplane, it is closed. If it contains no points in the hyperplane, it is open.

Concave and Convex Functions

It is useful to allow concave and convex functions to take on the values $\pm\infty$. Let $\mathbf{R}^\#$ denote the **extended real numbers**, $\mathbf{R}^\# = \mathbf{R} \cup \{-\infty, +\infty\}$.

Definition 2. *A function* $f : C \to \mathbf{R}^\#$ *defined on a convex set* C *of a vector space is* ***concave*** *if for all* $x, y \in C$ *and* $0 \leq \alpha \leq 1$

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

Convex functions *are defined defined similarly with the inequality reversed.*

Since f is convex iff $-f$ is concave, every result about concave (convex) functions has a corresponding result for convex (concave) functions with flipped inequalities. Most math books talk mostly about convex functions. Most econ texts talk mostly about concave functions.

Definition 3. *The* ***effective domain*** *of* f *is* $\operatorname{dom} f = \{x : f(x) > -\infty\}$. *A concave function* $f : C \to \mathbf{R}^{\#}$ *is* ***proper*** *if for all* $x \in C$ $f(x) < \infty$ *and* $\operatorname{dom} f \neq \emptyset$.

Every concave function $f : C \to \mathbf{R}$ extends to a concave function $\tilde{f} : \mathbf{R^n} \to \mathbf{R}$ where

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in C, \\ -\infty & \text{if } x \notin C. \end{cases}$$

1. Show that a function $f : \mathbf{R^n} \to \mathbf{R}$ is concave iff $\{(x, y) \in \mathbf{R^n} \times \mathbf{R} : y \leq f(x)\}$ is convex. This set is called the **subgraph** or **hypograph** of f.
2. State the equivalent fact for convex functions. The relevant set is called the **supergraph** or **epigraph**.

Hyperplanes and Separation Theorems

Definition 4. *A hyperplane* $[p = \alpha]$ ***separates*** *convex sets* A *and* B *iff for all* $x \in A$ *and* $y \in B$, $p \cdot x \leq \alpha$ *and* $p \cdot y \geq \alpha$ *(or vice versa). That is,* $A \subset [p \leq \alpha]$ *and* $B \subset [p \geq \alpha]$ *(or vice versa). The separation is* ***proper*** *iff there is some* $x \in A$ *and* $y \in B$ *for which* $p \cdot x \neq p \cdot y$.

Proper just means that the two convex sets do not lie in the hyperplane.

Definition 5. *Two convex sets* A *and* B *are* ***strongly separated*** *by* p *iff there is an* $\epsilon > 0$ *and* α *such that* $A \subset [p \leq \alpha]$ *and* $B \subset [p \geq \alpha + \epsilon]$ *(or vice versa).*

The key result for us is

Theorem 1 (Strong Separating Hyperplane Theorem)**.** *If K and C are non-empty disjoint convex subsets of* $\mathbf{R^n}$ *with K compact and C closed, then there is a $p \neq 0$ which strongly separates K and C.*

Strong separation means that there is a separating hyperplane containing neither set.

Primal and Dual Descriptions of Convex Sets

There are two ways to describe a closed convex set C. The **primal** description of C is the list of elements in C. The **dual** description of C is the set of closed half-spaces containing C.

3. Use the strong separating hyperplane theorem to show that if C is a closed convex set, then C is the intersection of the half-spaces containing it.

4. Give an example of two closed convex sets that cannot be strongly separated.

Definition 6. *The hyperplane $[p = \alpha]$ **supports** convex set C at $x \in C$ if $[p \geq \alpha] \supset C$ and $p \cdot x = \alpha$. In this event, $[p = \alpha]$ is a **supporting hyperplane** of C.*

5. Show that x is contained in the closed convex set C iff $x \in [p \geq \alpha]$ for all such half-spaces that contain C. This is the fundamental idea of duality.

6. Show that if C is a closed convex set, then the set of p such that $[p = p \cdot x^*]$ supports C at x^* is closed and convex. (This includes the possibility that this set is empty.)

Definition 7. *The **concave support function** of $C \subset \mathbf{R^n}$ is defined for all $p \in \mathbf{R^n}$ as*

$$e_C(p) = \inf\{p \cdot x, x \in C\}.$$

7. Show that $e_C(p)$ is concave.
8. Show that $e_C(p)$ is homogeneous of degree 1.
9. What does it mean if $e_C(p) = -\infty$? You can explain with a picture.
10. Show that $[p \leq \alpha]$ contains C iff $\alpha \geq e_C(p)$.

Continuity and Concavity

This material may be new.

Definition 8. *A real-valued function $f : X \to \mathbf{R}$ is* ***upper semi-continuous (usc)*** *on X if for all $\alpha \in \mathbf{R}$ the upper contour set $\{x : f(x) \geq \alpha\}$ is closed.*

The geometric intuition is that, for $X = \mathbf{R}$, f can jump up but not down. In textbooks you can find $\delta - \epsilon$ definitions and limit definitions that are less intuitive than this. One thing you can prove with them, though, is the following useful fact:

Theorem 2. *An extended real-valued function f is usc iff its subgraph is closed.*

Lower semi-continuity develops the same way, but with lower contour sets and closed supergraphs.

11. Prove that if f is both upper- and lower semi-continuous, its graph is closed.
12. Prove that if the functions in some collection $\{f_\alpha(x)\}$ are upper semi-continuous, than so is $f(x) = \inf_\alpha\{f_\alpha(x)\}$.

Of course, semi-continuity connects to continuity.

Theorem 3. *$f : X \to \mathbf{R}$ is both upper- and lower semi-continuous iff it is continuous.*

Finally, a definition of the most useful class of concave functions:

Definition 9. *A concave function f is* ***regular*** *if it is upper semi-continuous and proper.*

Supergradients

The good news about concave and convex functions is that, even though they may not be differentiable at some points in the domain, there are affine functions that act with with respect to the functions much like the way that derivatives act with respect to differentiable functions.

Definition 10. *A* ***supergradient*** *of* $f : C \subset \mathbf{R^n} \to \mathbf{R}$ *at* $x \in C$ *is a vector* $p \in \mathbf{R^n}$ *which satisfies the* ***supergradient inequality***

$$\textit{for all } y \in C,\ f(x) + p \cdot (y - x) \geq f(y). \tag{S}$$

The ***superdifferential*** *of* f *at* x *is the set*

$$\partial f(x) = \{p \in \mathbf{R^n} : p \textit{ satisfies } S\}.$$

13. Show that the hyperplane $(p, -1)$ supports the subgraph of f at $(x^*, f(x^*))$

14. The concave indicator function of a closed convex set C is

$$I_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ -\infty & \text{otherwise.} \end{cases}$$

    Show that the hyperplane $[p = p \cdot x]$ supports C at x iff $p \in \partial I_C(x)$.

15. Let $f(x) = \sqrt{x}$ on $\mathbf{R}_+$. For each $x \in \mathbf{R}_+$, describe $\partial f(x)$.

16. Show that if $f : C \to \mathbf{R}$ is a proper concave function and $0 \in \partial f(x^*)$, then x^* is a global maximum of f on C.