The Two-Sector Model

1. Find cost functions for the following production functions;
   (a) Cobb-Douglas.
   (b) CES, $f(k,l) = (ak^r + bl^r)^{1/r}$.
   (c) linear, $f(k,l) = ak + bl$.
   (d) Leontief, $f(k,l) = \min\{ak, bl\}$.
   (e) von Thünen's production function, $f(k,l) = (1 - e^{-ak})(1 - e^{-bl})$.

2. Suppose in a two-country world, countries A and B, that f_A and f_B are the usual neoclassical production functions of capital and labor, with Inada conditions.
   (a) Show that alll revenues (national product) are distributed to the factors.
   (b) Suppose that p_A increases. Under what conditions on f_A and f_B will the capital share of national product increase?

3. For a country with an endowment in the interior of the cone of diversification, derive and prove a result on the effects of a small increase in the quantity of a factor on output.

4. Here is another two-sector model. Sector 1 produces investment goods (capital goods). Sector 2 produces consumption goods. Each sector is characterized by a neoclassical production function (strictly concave, C^2, Inada conditions at 0) with constant returns to scale. Write $Y_i = F_i(K_i, L_i)$ for output in sector i as a function F_i of capital K_i and labor L_i employed in sector i.
   (a) Rewrite these relationships in terms of the output/labor and capital/labor ratio: $y_i = f_i(k_i)$ where $y_i = Y_i/L_i$, etc. What properties do the foregoing assumptions imply for the f_i?
   (b) Let w and r denote the equilibrium prices of capital, and P_i the price of output i. Equilibrium in this model requires for $i = 1, 2$

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\begin{aligned}
Y_i &= F_i(K_i, L_i) \\
P_i \frac{\partial F_i}{\partial K_i} &= r & \qquad P_i \frac{\partial F_i}{\partial L_i} &= w \\
K_1 + K_2 &= K & \qquad L_1 + L_2 &= L \\
P_1 Y_1 &= rK & \qquad P_2 Y_2 &= wL
\end{aligned}
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   Interpret these equations, including the demand conditions (4).
   (c) Rewrite these conditions in terms of the aggregate capital/Labor ratio $k = K/L$, the output/labor ratios y_i, the labor shares $l_i = L_i/L$ and the wage/rent ratio $\omega = w/r$.
   (d) Compute $dk_i/d\omega$ and use this to prove that the capital/labor ratio in each sector is uniquely determined by the wage/rent ratio.

(e) Find an equation that implicitly defines the equilibrium wage-rent ration ? in terms of the capital-labor ratio.