Transferable Utility Matching

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Introduction

Marshall had fishing hooks in mind.

But markets are not just about quantity adjustment.

What Do Markets Do?

- Markets sort these people.
- Markets match them into firms.
- Markets aggregate information, thereby reducing transaction costs.

A Labor Market for Lawyers

The market contains "workers" and "firms." Workers and firms are matched together, one-to-one. Utility is transferable among workers and firms. Each matched, if formed will generate surplus. We ask:

- Characterize optimal matches?
- Can they be decentralized by a market-like arrangement?
- How is the surplus allocated among workers and firms?
- Is there a mechanism that will implement the market solution?

Classic article: Shapley and Shubik. 1971. "The Assigment Game I: The Core." *International Journal of Game Theory* 1(1): 111-30.

Applications of the Model

Labor markets

- Sattinger, M. 1993. "Assignment models of the distribution of earnings." *Journal of Economic Literature* 31:2, 331–80.
- Chade H., Eeckhout J., Smith L. 2017. "Sorting through search and matching models in economics." *Journal of Economic Literature* 55(2): 493–544.
- Eeckhout, J. 2018. "Sorting in the labor market." *Annual Review of Economics* 10: 1–29.

International Trade

- Costinot, A. and Vogel, J. 2015. "Beyond Ricardo: Assignment models in international trade." I: 31–62.

The Model

L workers match with F firms. Each worker can match with at most one firm; each firm can match with at most one worker.

- $\mathcal{L}$ Workers.
- $\mathcal{F}$ Firms.
- v_{lf} The surplus from matching worker l with firm f.
- π_f Profit of firm f.
- w_l Wage of worker l.
- x A matching. x_{lf} is 1 if l is matched with f, 0 otherwise.

Optimality

A matching is optimal if it maximizes total surplus.

$$
\begin{aligned}
v(\mathcal{L} \cup \mathcal{F}) = \max_{l,f} \; & v_{lf} x_{lf} \\
\text{s.t.} \quad & \sum_f x_{lf} \leq 1 \text{ for all } l \in \mathcal{L}, \\
& \sum_l x_{lf} \leq 1 \text{ for all } f \in \mathcal{F}, \\
& x_{lf} \in \{0, 1\} \text{ for all } l \in \mathcal{L}, f \in \mathcal{F}.
\end{aligned}
\tag{1}
$$

A matching x is optimal iff it solves this optimization problem.

Example

$$\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{\mathcal{F}} \\
 & & 1 & 2 & 3 \\
\cline{3-5}
 & 1 & 1 & 8 & 3 \\
\cline{3-5}
\mathcal{L} & 2 & 3 & 1 & 8 \\
\cline{3-5}
 & 3 & 8 & 3 & 1 \\
\cline{3-5}
 & 4 & 7 & 7 & 7 \\
\cline{3-5}
\end{array}$$

The optimal match is $1 \leftrightarrow 2, \quad 2 \leftrightarrow 3, \quad 3 \leftrightarrow 1.$

Equilibrium

A payoff is a vector $(w_l, \pi_f)_{l,f \in \mathcal{L} \cup \mathcal{F}} \geq 0$.

An allocation is a matching-payoff pair (x, w, π) such that:

- if $x_{lf} = 1$,then $w_l + \pi_f = v_{lf}$,
- if $x_{lf} = 0$ for all f, then $w_l = 0$,
- if $x_{lf} = 0$ for all l, then $\pi_f = 0$.

An allocation (x, w, π) is stable if no currently unmatched worker-firm pair can increase their total surplus by matching to each other. That is,

$$\text{if } x_{lf} = 0, \text{then } w_l + \pi_f \geq v_{lf}.$$

An LP Relaxation

Consider the LP

$$
\begin{aligned}
v_P(\mathcal{L} \cup \mathcal{F}) = \max_{l,f} \; & v_{lf} x_{lf} \\
\text{s.t.} \quad & \sum_f x_{lf} \le 1 \text{ for all } l \in \mathcal{L}, \\
& \sum_l x_{lf} \le 1 \text{ for all } f \in \mathcal{F}, \\
& x_{lf} \ge 0 \text{ for all } l \in \mathcal{L}, f \in \mathcal{F}.
\end{aligned}
\tag{2}
$$

The set C of all vectors satisfying the constraints is a convex polytope, the "fractional matchings".

Theorem (Birkhoff-von Neuman). x is a vertex of C iff for all l, f $x_{lf} \in \{0, 1\}$.

Corollary. x^* is an optimal matching for (1) iff it is a basic optimal solution to the LP (2).

The Dual LP

$$
\begin{aligned}
v_D(\mathcal{L} \cup \mathcal{F}) = \min_{\pi, w} \sum_{lf} w_l + \pi_f & \\
\text{s.t.} \quad \begin{array}{c} w_l + \pi_f \geq v_{lf} \text{ for all } l \in \mathcal{L}, f \in \mathcal{F}, \\ w_l, \pi_f \geq 0 \text{ for all } l \in \mathcal{L}, f \in \mathcal{F}. \end{array} &
\end{aligned}
\tag{3}
$$

- The dual has a solution (w^*, π^*) and $\sum_{lf} w_l^* + \pi_f^* = \sum_{lf} v_{lf} x_{lf}^*$.
- If $x_{lf}^* = 1$ then $w_l^* + \pi_f^* = v_{lf}$.
- If $l \in \mathcal{L}$ is unmatched, then $w_l = 0$.
- If $f \in \mathcal{F}$ is unmatched, then $\pi_f = 0$.

Theorem. (x^*, w^*, π^*) is a stable allocation iff x^* is an optimal matching and (w^*, π^*) solves the dual lp.

Example

Consider the surplus matrix

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\mathcal{F}} \\ & & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} \\ \cline{3-4} \multirow{2}{*}{$\mathcal{L}$} & 1 & 10 & 9 \\ \cline{3-4} & 2 & 9 & 3 \\ \cline{3-4} \end{array}$$

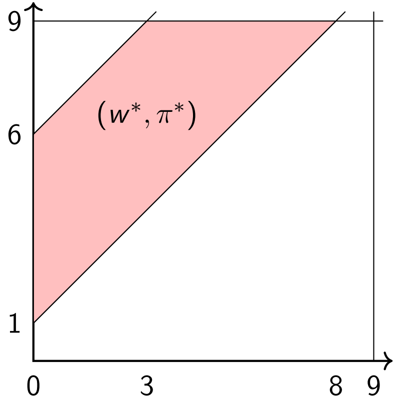
The optimal match is $1 \leftrightarrow 2$, $2 \leftrightarrow 1$, with a surplus of 18. The dual constraints are

$$\begin{aligned} w_1 + \pi_1 &\geq 10 \\ w_1 + \pi_2 &\geq 9 && (*) \\ w_2 + \pi_1 &\geq 9 && (*) \\ w_2 + \pi_2 &\geq 3. \end{aligned}$$

Example

The marked constraints are binding, so substitute for π_1 and π_2.

$$w_1 - w_2 \geq 1$$
$$w_1 - w_2 \leq 6$$



Example

$$\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{\mathcal{F}} \\
 & & 1 & 2 & 3 \\
\hline
 & 1 & 1 & 8 & 3 \\
\hline
\mathcal{L} & 2 & 3 & 1 & 8 \\
\hline
 & 3 & 8 & 3 & 1 \\
\hline
 & 4 & 7 & 7 & 7 \\
\hline
\end{array}$$

Aside from nonnegativity there are 12 constraints in the dual.

- The primal constraint on worker 4 does not bind, so $w_4 = 0$.
- $w_4 + \pi_i \geq 7$, so $\pi_i \geq 7$.
- $\pi_{i+1} = 8 - w_i$.
- $0 \leq w_i \leq 1$.

The unemployed worker 4 constrains everyone else's wages.

The Core

No worker-firm pair can break off and do better on their own. What about larger coalitions of workers and firms?

Let $S \subset \mathcal{L} \cup \mathcal{F}$ be a set of workers and/or firms. If $S \subset \mathcal{L}$ or $S \subset \mathcal{F}$ let $v_P(S) = 0$. Otherwise, define the total surplus S can earn for itself.

$$
\begin{aligned}
v_P(S) = \max_{l,f} & \; v_{lf} x_{lf} \\
\text{s.t.} \quad & \sum_f x_{lf} \leq 1 \text{ for all } l \in S, \\
& \sum_l x_{lf} \leq 1 \text{ for all } f \in S, \\
& x_{lf} \geq 0 \text{ for all } l, f \in S.
\end{aligned} \tag{4}
$$

If $\sum_{l,f \in S} w_l + \pi_f < v_P(S)$, then S can improve itself by breaking away.

The Core

The matching problem defines a transferable utility game. A payoff is in the core of the matching game if no subset S of individuals can improve themselves by breaking away.

Theorem. Any stable payoff is a core payoff.

Proof. Consider without loss of generality the coalition containing workers 1 through k and firms 1 through k. Suppose that the optimal matching in the coalition matches each worker i with firm i. For any stable payoff (w^*, π^*) for the entire group, $w_i^* + \pi_i^* \geq v_{ii}$, so

$$\sum_{i=1}^{k} w_i^* + \pi_i^* \geq \sum_{i=1}^{k} v_{ii} = v_P(S),$$

so coalition S cannot improve upon any stable payoff. $\square$

This would be the moment to discuss transferable utility games more generally.

Lattices

A partially-ordered set $(X, \succcurlyeq)$ is a set with a reflexive, transitive, and antisymmetric binary relation $\succcurlyeq$.

$x \in X$ is an upper bound for $A \subset X$ if $x \succcurlyeq y$ for all $y \in A$. x is a supremum of A if it is an upper bound for a and there is no upper bound y for A with $x \succ y$. Similarly for lower bounds.

A lattice is a poset $(X, \succcurlyeq)$ in which each pair of elements $x, y \in X$ has a supremum $x \vee y \in X$ and an infimum $x \wedge y \in X$.

A lattice is complete if every subset A of X has both a lub and a glb in X.

$A \subset X$ is as big as $B \subset X$ in the strong set ordering, $A \sqsupseteq B$ if for all $x \in A$ and $y \in B$, $x \vee y \in A$ and $x \wedge y \in B$.

The Lattice Property

Let P denote the set of stable payoffs. Define $(w, \pi) \succcurlyeq (w', \pi')$ if for all l $w'_l \geq w''_l$ and for all f $\pi'_f \leq \pi''_f$, each in the usual vector order.

Theorem. $(P, \succcurlyeq)$ is a complete lattice.

Consequence: There is a unique least wage payoff (w', π') and a unique greatest wage payoff (w'', π''). They are best and worst, respectively, for firms.

The Lattice Property

Proof. Choose two payoff vectors $p' = (w', \pi')$ and $p'' = (w'', \pi'')$. First we show that $p' \vee p''$ satisfies the dual constraints.

$$p' \vee p'' = (\max\{w'_l, w''_l\}, \min\{\pi'_f, \pi''_f\})_{l,f \in \mathcal{L} \cup \mathcal{F}}.$$

Then for all lf pairs,

$$\begin{aligned} w'_l &\geq v_{lf} - \pi'_f \\ w''_l &\geq v_{lf} - \pi''_f. \end{aligned} \tag{†}$$

and so

$$\begin{aligned} \max\{w'_l, w''_l\} &\geq \max\{v_{lf} - \pi'_f, v_{lf} - \pi''_f\} \\ &= v_{lf} - \min\{\pi'_f, \pi''_f\} \end{aligned}$$

So $p' \vee p'' \geq v_{lf}$. A similar argument holds for $p' \wedge p''$.

The Lattice Property

Finally, we show that $p' \vee p''$ satisfies complementary slackness. Feasible solutions satisfying complementary slackness are optimal solutions for the dual problem. Suppose $x_{lf} = 1$. Then the equations (†) hold with equality, and so the conclusion holds with equality as well. Again, a similary argument applies to $p' \wedge p''$. This proves that $(P, \succeq)$ is a lattice.

The Lattice Property

We have to show that $(P, \succeq)$ is complete. Let $A \subset P$ be a set of payoffs. The natural sup candidate is $\bar{p}$ such that $\bar{w}_l = \sup\{w_l : p \in A\}$ and $\bar{\pi}_f = \inf\{\pi_f : p \in A\}$. Then $\bar{p} = \sup\{p : p \in A\}$. For $(P, \succeq)$ to be complete, $\bar{p} \in P$.

For all $\epsilon > 0$ and for each lf pair there is a p such that $w_l \leq \bar{w}_l < w_l + \epsilon$ and $\pi_f \geq \bar{\pi}_f > \pi_f - \epsilon$. Then

$$v_{lf} - \epsilon \leq w_l + \pi_f - \epsilon \leq \bar{w}_l + \bar{\pi}_f$$

Let $\epsilon \to 0$ to see that $\bar{w}_l + \bar{\pi}_f \geq v_{lf}$, satisfying the dual constraint. If lf is part of the optimal match,

$$\bar{w}_l + \bar{\pi}_f \leq w_l + \epsilon + \pi_f = v_{lf} + \epsilon$$

Letting $\epsilon \to 0$, $\bar{w}_l + \bar{\pi}_f \leq v_{lf}$. Thus complementary slackness is satisfied, and so $\bar{p} \in P$. $\square$

Positive and Negative Assortative Matching

PAM

NAM

Positive Assortative Matching

Suppose we are given partial orders $\succ_l$ on workers and $\succ_f$ on firms. For instance, $l' \succ_l l''$ might mean that worker l' is more skilled than is worker l'', and $f' \succ_f f''$ might mean that higher skill levels are more productive in firm f' than in firm f''.

Theorem. Suppose that if $l' \succ_l l''$ and $f' \succ_f f''$, then $v_{l'f'} - v_{l''f'} > v_{l'f''} - v_{l''f''}$. Then it cannot be the case that $l' \leftrightarrow f''$ and $l'' \leftrightarrow f'$.

- The idea of matching bigger with bigger is called positive assortative matching. It came to prominence first in Becker (1973) on marriage.
- The property that the v differences in l are increasing in f is called increasing differences.

Positive Assortative Matching

Proof. If

$$v_{l'f'} - v_{l''f'} > v_{l'f''} - v_{l''f''},$$

then

$$v_{l'f'} + v_{l''f''} > v_{l'f''} + v_{l''f'}.$$

So matching l' to f' and l'' to f'' would increase surplus. $\square$

Comparative Statics

How do wages and profits change with the v_{lf}? Who gains and who loses?

We use the framework of monotone comparative statics.

Choose l and f, and order (w_l, π_f) pairs as follows:

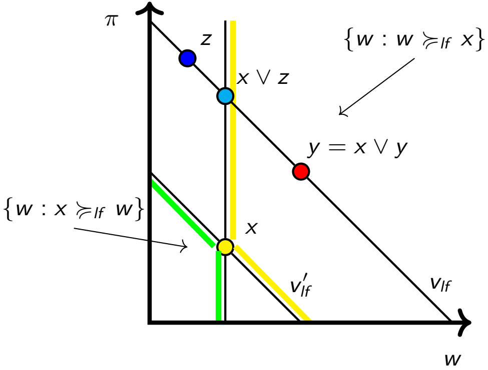
$$(w, \pi) \succcurlyeq_{lf} (w', \pi') \text{ iff } w_l + \pi_f \geq w'_l + \pi'_f \text{ and } w_l \geq w_{l'}.$$

Order remaining wages and profits with the usual $\geq$ order.

Finally, order $\mathbf{R}_+^{\mathbf{L}} \times \mathbf{R}_+^{\mathbf{F}}$ with the product order: $p \succcurlyeq p'$ iff

$$\text{for all } l'f' \neq lf,\ (w_{l'}, \pi_{f'}) \geq (w_{l'}, \pi_{f'}) \text{ and } (w_l, \pi_f) \succcurlyeq_{lf} (w'_l, \pi'_f).$$

Comparative Statics



Comparative Statics

Suppose v_{lf} increases to v'_{lf}. There are three cases: i) lf is an in-the-money match; ii) lf is not in the money, and remains out of the money; and iii) lf is not in the money, but becomes so. But there are really only two cases. Case iii) can be decomposed into a change in lf over the range where there is an optimal matching in which l is not matched with f, and a change over the range where lf is part of an optimal matching. Th two ranges intersect at a point where there are (at least) two optimal matches, and all such matches have the same value.

Case i) is easily dispensed with. If lf is part of an optimal matching, and v_{lf} increases, it remains so. (Otherwise there is a matching which pays off more than the original optimal match, does not match lf, and so would have been feasible and have the same payoff at the initial constraints.) The set of dual solutions increases the payoffs available to lf, and leaves everything else unchanged.

Comparative Statics

Take any dual solution to the new problem. Every other pair must be dividing up the value of their match, so the set of allocations of these surpluses in the old and new problem must be identical. And lf must divide their surplus v'_{lf}, so their payoff set has strongly increased.

Now consider an out-of-the-money lf pair, and let $\sigma(l)$ denote ls optimal match. Suppose $v'_{lf} > v_{lf}$, $\sigma(l) \neq f$ and σ is optimal on the interval $[v_{lf}, v'_{lf}]$, ceteris paribus. Consider the minimal wage for l and the maximal profit for $\sigma(l)$:

$$\underline{w}_l + \bar{\pi}_{\sigma(l)} = v_{l\sigma(l)}.$$

Comparative Statics

Lemma. If $\underline{w}_l \neq 0$, there is a $f \neq \sigma(l)$ such that

$$\underline{w}_l + \bar{\pi}_f = v_{lf},$$

and similarly for $\underline{\pi}_f$.

Proof. $(\underline{w}_l, \bar{\pi}_f)_{lf \in \mathcal{L} \cup \mathcal{F}}$ is a dual-optimal payoff. Suppose the claim is false. Then we have $\underline{w}_l + \bar{\pi}_f \geq v_{lf} + \epsilon$ for some $\epsilon > 0$ and all $f \neq \sigma(l)$. Modifying the payoff by letting $w'_l = \underline{w}_l - \epsilon'$ and $\pi'_{\sigma(l)} = \pi_{\sigma(l)} + \epsilon'$ is feasible for any $0 < \epsilon' < \epsilon$, the payoff has the same value, so it too solves the dual, contradicting the minimality of $\underline{w}_l$. $\square$

Call this constraint the opportunity constraint for l. There is also an opportunity constraint for matching f with $\sigma^{-1}(f)$.

Comparative Statics

Theorem. Suppose that lg is the unique opportunity constraint for l. An increase in v_{lg} raises $\underline{w}_l$ and decreases $\bar{\pi}_{\sigma(l)}$. A decrease in v_{lg} lowers $\underline{w}_l$ and increases $\bar{\pi}_{\sigma(l)}$.

Proof. Replace v_{lg} by $v'_{lg} > v_{lg}$ such that σ remains an optimal match. Then the binding opportunity constraint on w_l is tighter, so w_l increases. Replace v_{lg} by $v'_{lg} < v_{lg}$ such that σ remains an optimal match. Then there is no binding opportunity constraint for l, and the argument of the Lemma's proof shows that the new greatest lower bound $\underline{w}'_l$ on w_l is less than $\underline{w}_l$, and that $\bar{\pi}'_{\sigma(l)} > \bar{\pi}_{\sigma(l)}$. $\square$

Similarly, if lg is the unique opportunity constraint for f. Then raising v_{lg} lowers $\underline{\pi}_f$ and raises $\bar{w}_{\sigma^{-1}(f)}$.

So if we increase v_{lf} from a very low value for $f \neq \sigma(l)$, nothing happens until lf becomes the opportunity constraint for l. Then $\underline{w}_l$ rises until it becomes optimal to assign l to f. Then $\underline{w}_l$ holds constant. Along this same trajectory, $\bar{\pi}_{\sigma(l)}$ holds constant, then falls, and then holds constant when it is no longer optimal to match l with $\sigma(l)$.

The Assignment Problem

The assignment problem is a one-sided version of the matching market. Example: The objective is to match individuals to positions. It can be set up as an lp in the same way as the matching market. There are two versions:

- Position constraints. The "profits" to positions are what the individuals pay for the position, the "wage" is remaining surplus, that they keep.
- No constraints. Model this as more positions of each type than there are agents in the economy. Then the position price will be 0 and individuals keep all the surplus. Here the focus is on self-selection and the match. An example of this is the Roy (1951) model.