Markets and Uncertainty

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Contents

- Specialize the abstract Arrow–Debreu model to interpret it as trading under uncertainty.
- Interpret commodities, preferences, endowments, budget sets, etc., to apply to situations with uncertainy.
- Examine alternative market specifications.

What questions do we want to ask in these models?

- Comparative statics of preference parameters — risk, ambiguity aversion.
- Welfare properties of markets.
- The structure of equilibrium — the composition of prices.

Representation of Uncertainty

- Commodities are distributions on outcomes. Prices are linear functionals of distributions — integration of a function.
- Commodities are random variables — measurable functions of *states*. Prices are linear functionals of measurable functions — measures on the state space.

The State Preference Model

O The *outcome space.*

S The finite *state space.*

C The set of *acts* — functions from states to outcomes.

$\succcurlyeq$ A preference relation on C.

Examples

$$U(f) = \min_s u\big(f(s)\big) \qquad U(f) = \sum_s u\big(f(s)\big)\mu(s)$$

$$U(f) = \min_{\mu \in P} \sum_s u\big(f(s)\big)\mu(s)$$

$$U(f) = \frac{1}{1-\gamma} \int_P \left(\sum_s u\big(f(s)\big)\mu(s) \right)^{1-\gamma} dp(\mu)$$

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Quiz 1

- Suppose the state space is finite and that $O = \mathbf{R}^L_+$. In each caser, determine what conditions on u, μ, etc., guarantee the existence of equilibrium in the private ownership economy.

- Minimax Regret Demand. On each budget set $B(p, y) = \{x : \sum_s p_s \cdot x(s) \leq y\}$, let $v(s, p, y) = \max_{x \in B(p,y)} u(x(s))$. State s regret of act x is then $r(x, s, p, y)$ which is $v(s, p, y) - u(x(s))$. Now define max regret $U(x, p, y) = \max_s r(x, s, p, y)$. Demand on the budget set is:

$$d(p, y) = \operatorname*{argmin}_{x \in B(p,y)} U(x, p, y).$$

  Interpret this. Under what conditions on u will demand be such that aggregate excess demand satisfies assumptions 1,2,3′, and 4 for the existence of a competitive equilibrium?

The 2-Period Exchange Economy

- I traders live for two periods; $t = 0, 1$.
- State $s \in \mathcal{S}$ is revealed at the beginning of the second period.
- L physical commodities available at date 0 and states 1 through S. The consumption set for each trader is $\mathbf{R}_+^{L(S+1)}$.
- Each trader has a strictly positive endowment vector $\omega^i \gg 0$ in $\mathbf{R}_+^{L(S+1)}$.
- Each trader has beliefs π^i and concave and increasing date t payoff functions $u_t^i : \mathbf{R}_+^L \to \mathbf{R}$.

$$U^i(x^i) = u_0^i(x_0^i) + \sum_{s=1}^{S} \pi_s^i u_1^i(x_s^i)$$

The Arrow-Debreu Model

At date 0 traders trade current consumption bundles and contracts promising the delivery of x_{sl} units of good l if state s occurs in date 1.

- Prices are $\phi = (\phi_0, \ldots, \phi_S) \in \mathbf{R}^L_+ / 0$.
- The Arrow-Debreu budget set is

$$B_{AD}(\phi, \omega^i) = \left\{ y \in \mathbf{R}_+^{L(S+1)} : \sum_{s=0}^{S} \phi_s (y_s - \omega_s^i) = 0 \right\}.$$

Equilibrium in the AD model is a price ϕ and allocation x such that

a) Each trader i is maximixing $U^i(x^i)$ on $B_{AD}(\phi, \omega^i)$;

b) For all s and l, $\sum_i x^i_{sl} - \omega^i_{sl} = 0$.

The Arrow-Debreu Model

We learn from the general analysis:

- Equilibrium exists.
- The First and Second Welfare Theorems apply

What is the meaning of Pareto Optimality here?

Quiz 2

- Draw an Edgeworth box for an economy with two expected utility maximizers, one commodity, and two states of the world. Neither preferences nor endowments need be the same, but the aggregate endowment of the economy is the same in both states of nature. Suppose first that they have identical beliefs. Second, suppose they have different beliefs.
- With expected utility preferences and one good in each state, what is the MRS for any diagonal consumption bundle?

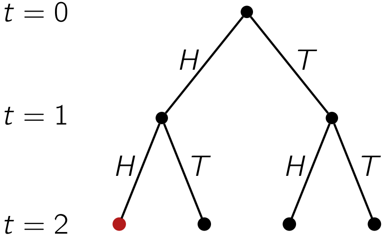
The Radner Model

- Traders can trade physical commodities in date 0 and state s spot markets, and J assets.
- Asset j is a promise to pay to its holder a_s^j units of the numeraire good 1 in state s. The asset return matrix is A, with rows A_s.
- The vector of state-contingent returns from portfolio $z \in \mathbf{R}^J$ is Az.
- The set of feasible portfolios is unbounded. $z^j < 0$ is a sale/short position on the numeraire payouts; $z^j > 0$ is a purchase/long position on the numeraire payouts.

Prices in the Radner model are spot prices $(p_0, \ldots, p_S) \in \mathbf{R}_+^{L(S+1)}$ and asset prices $q \in \mathbf{R}^J$.

The Radner Model

The Date-Event Tree



Each node is a spot market. Each edge is a state. The red node is the spot market after 2 H states are realized. People trade goods at spot markets. THey trade assets between spot markets.

The Radner Model

The Radner budget set is

$$B_R(p, q, \omega^i) = \left\{ \begin{array}{c} (y, z) \in \mathbf{R}_+^{L(S+1)} \times \mathbf{R}^J \text{ s.t.} \\ p_0 y_0 + qz = p_0 \omega_0^i \\ \text{for } s \in S,\ p_s y_s = p_s \omega_s^i + p_{s1} A_s z \end{array} \right\}.$$

Equilibrium in the Radner model is a vector of spot prices p, asset prices q, a commodity allocation x and an asset allocation z such that

a) Each trader i is maximizing $U^i(x^i)$ on $B_R(p, q, \omega^i)$;
b) For all s and l, $\sum_i x_{sl}^i - \omega_{sl}^i = 0$;
c) $\sum_j z^j = 0$.

The No-Arbitrage Condition

If there is a portfolio with a semi-positive return vector, then some spot-market budget sets are unbounded. Equilibrium requires is that no such portfolio exists. This condition may restrict asset prices.

Let

$$M = \begin{bmatrix} -q \\ A \end{bmatrix}$$

with column space $< M >$.

The no arbitrage condition is satisfied if there is no portfolio z such that $Mz > 0$. That is, $< M > \cap \mathbf{R}_+^{S+1} = \{0\}$.

The No-Arbitrage Theorem

Theorem. The no-arbitrage condition is satisfied iff there is a $\tilde{\pi} \gg 0$ such that $\tilde{\pi} M = 0$.

Let $P(q) = \left\{\pi \in \mathbf{R}^S_{++} : (1, \pi) \cdot M = 0\right\}$. $\pi \in P(q)$ iff

$$q^j = \pi_1 a^j_1 + \cdots + \pi^S a^j_S.$$

Any member of $P(q)$ is called a state price vector.

$\dim P(q) = S - \mathrm{rank}\, A$. So π will be unique iff $\mathrm{rank}\, A = S$. In this case, markets are said to be complete.

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Proof of the No-Arbitrage Theorem

Let Δ denote the set of $y \in \mathbf{R}_+^{S+1}$ such that $\sum_n y_n = 1$. $Mz > 0$ has a solution iff $Mz \in \Delta$ has a solution.

Lemma. If $A, B \subset \mathbf{R}^m$ are convex, A closed and B compact, then there is a π that separates them, that is, $\sup_{a \in A} \pi a < \inf_{b \in B} \pi b$.

(Sufficiency) If the condition of the theorem is satisfied, then $\pi M z = 0$ for all z. Since $\pi \gg 0$, $Mz \not> 0$.

Proof of the No-Arbitrage Theorem

(Necessity) Let $D = \{y : y = Mz, z \in \mathbf{R}^J\}$. The no-arbitrage condition implies that $\Delta \cap D = \emptyset$. Thus there is a π that separates them.

To apply the lemma, take $A = D$ and $B = \Delta$. If for some s $\pi_s \leq 0$ then $\sup_{d \in D} \pi d \geq 0 > \inf_{\delta \in \Delta} \pi \delta$, a contradiction. Thus $\pi \gg 0$.

If $\pi d \neq 0$ for some $d \in D$, then $\pi M z \neq 0$ for some $z \in \mathbf{R}^J$. If so, $\sup_{d \in D} \pi d = \sup_{z \in \mathrm{R}^J} \pi M z = +\infty$, contradicting the separation property. Thus $\pi M z = 0$ for all $z \in \mathbf{R}^J$, so $\pi M = 0$. □

Implications of No Arbitrage

Redundant Assets

Suppose asset $A^j = \sum_{k \neq j}^{K} \alpha_k A^k$. Each $q_k = \pi A^k$, so

$$q_j = \pi A^j = \pi \sum_{k \neq j} \alpha_k A^k = \sum_{k \neq j} \alpha_k \pi A^k = \sum_{k \neq j} \alpha_k q_k.$$

Quiz 3

1. Suppose there are two states and three assets, and

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

   Find all state prices, and asset prices such that the no-arbitrage condition is satisfied.

2. Suppose there are three state and two assets.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$$

   Find all state prices, and asset prices such that the no-arbitrage condition is satisfied.

3. Under what circumstances do asset prices uniquely define the state price vector?

Arrow Securities

Suppose A is the $S \times S$ identity matrix. No arbitrage implies $q \gg 0$. Then $\pi = q$.

π_s is the price of a unit of numeraire in state s.

The securities that pay off 1 in some state s and 0 otherwise are called Arrow securities.

Arrow's (other) Theorem

Theorem. Suppose that $\operatorname{rank} A = S$.

a) Suppose Radner prices are (p, q), with state prices π, and define $\phi_0 = p_0$; for $s \geq 1$, $\phi_{sl} = \pi_s p_{sl}/p_{s1}$. $(x, z) \in B_R(p, q, \omega^i)$ iff $x \in B_{AD}(\phi, \omega^i)$.

b) Suppose A-D prices are ϕ, and define $p_0 = \phi_0$; $\pi_s = \phi_{s1}$; $p_{sl} = \phi_{sl}/\phi_{s1}$, and $q = \pi A$. $x \in B_{AD}(\phi, \omega^i)$ iff there is a z such that $(x, z) \in B_R(p, q, \omega^i)$.

Implications:

- When markets are complete, Radner and Arrow-Debreu equilibrium commodity allocations are identical.
- Prices of any one equilibrium can be derived from the other.

Quiz 4

Suppose in an economy with two physical goods that a consumer has endowment $(1, 1)$ at time 0, $(1, 0)$ in state 1 and $(0, 2)$ in state 2. Arrow-Debreu prices at time 0 are $(1, 1)$, $(1, 2)$ in state 1 and $(1, 1)$ in state 2.

- Check that the consumption bundle x which contains $(1, 1)$ at time 0, $(1/2, 1/2)$ in state 1 and $(3/4, 3/4)$ in state 2 is affordable.
- Suppose now a Radner economy with the same endowments and the equivalent Radner prices. Suppose asset 1 is a bond that pays off 1 unit of numeraire in each state, and asset 2 pays off 1 unit in state 1 and 0 otherwise. Find the asset prices, and the portfolio z that makes the consumption bundle (x, z) budget-feasible.

Equilibrium FOC

λ_s^i is the multiplier for constraint s, $s = 0, \ldots, S$.

$$\text{for all } s \text{ and } l,\ D_{sl} U^i(x^i) - \lambda_s p_{sl} = 0$$

$$\text{for all } j,\ -\lambda_0^i q_j + \sum_j \lambda_s^i a_s^j = 0$$

budget feasibility

$$x \gg 0 \qquad \lambda \gg 0$$

The Stochastic Discount Factor

Suppose expected utility: There is a probability distribution ρ and a function u such that $U(x) = u(x_0) + \sum_s \pi_s u(x_s)$.

The state s intertemporal MRS (IMRS^i_s) is $D_1 u^i(x^i_{s1})/D_1 u^i(x^i_{01})$.

From trader i's FOC:

$$q_j = \sum_s \rho_s \mathrm{IMRS}^i_s a^j_s \equiv \mathrm{E}_\rho \, IMRS^i a^j.$$

That is, $(\rho_s \mathrm{IMRS}^i_s)_{s\in\mathcal{S}} \in P(q)$.

In finance, the function $s \to IMRS^i_s$ is called a stochastic discount factor.

Quiz 5

- Prove the relationship between Radner and Arrow-Debreu budget sets.

- Derive the asset price equation on slide 22.

- Consider an economy with 2 states and a single consumption good. $U^i(x) = \log x_0 + \pi_1^i \log x_1 + \pi_2^i \log x_2$. For all i and $s \geq 0$, $\omega_s^i = 1$.
  - Derive the first-order conditions for the case of a single asset which is a bond, paying off 1 in each state $s \geq 1$.
  - What is the equilibrium consumption and portfolio allocation?
  - Suppose that $\pi_1^1 \neq \pi_1^2$. Argue intuitively why equilibrium is not Pareto optimal.
  - Prove it is not optimal from the first order conditions.

Welfare Economics

Arrow's other Theorem implies that if markets are complete, then the first and second welfare theorems hold for Radner equilibria. What can go wrong when markets are incomplete?

The FOC with Arrow securities:

$$D_{sl} U^i(x^i) - \lambda_s^i p_{sl} = 0,$$

$$-\lambda_0^i q^s + \lambda_s^i = 0,$$

etc.

Welfare Economics

$$\frac{D_{sl}U^i(x^i)}{D_{sm}U^i(x^i)} = \frac{p_{sl}}{p_{sm}}$$

Marginal rates of substitution between goods in the same state are equal.

$$\frac{D_{sl}U^i(x^i)}{D_{tm}U^i(x^i)} = \frac{\lambda_s^i}{\lambda_t^i}\frac{p_{sl}}{p_{tm}}$$

$$\frac{\lambda_s^i}{\lambda_t^i} = \frac{q_s}{q_t}$$

Marginal rates of substitution between goods in different states equal a constant times the marginal rate of substitution between the numeraires in the different states. But they are also equal.

Welfare Economics

Suppose only a bond that pays off 1 in each state:

$$D_{sl}U^i(x^i) - \lambda_s^i p_{sl} = 0,$$

$$-\lambda_0^i q^1 + \sum_s \lambda_s^i = 0,$$

etc.

Welfare Economics

The vector $\left(\frac{\lambda_1^i}{\lambda_0^i}, \ldots, \frac{\lambda_S^i}{\lambda_0^i}\right)$ is a state price vector.

The first order conditions do not uniquely determine state prices!

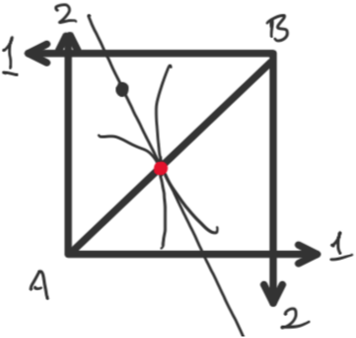
Marginal rates of substitution between goods in different states are not necessarily equal.

Quiz 1 Answers

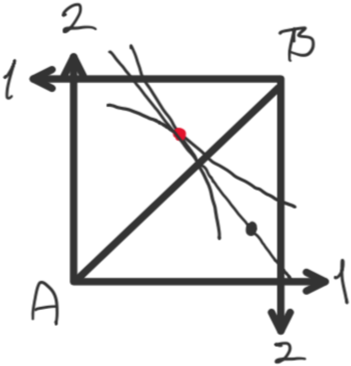
1. (Maxmin): u is continuous, strictly increasing. (EU): u is continuous, concave, increasing. Convince yourself that the sum of two quasi-concave functions need not be quasi-concave.) Maxmin EU): u is continuous, concave, increasing. P is closed. (Smooth Ambiguity): u is continuous, concave, increasing. $\gamma \geq 0, \neq 1$.
2. v is the maximal utility achievable in any state Regret in state s is the loss from the maximal achievable utility at s. U is maximum regret. Sufficient conditions are that u is continuous, strictly increasing and strictly quasi-concave, and indifference curves do not intersect the axes. Does minimax regret demand satisfy WARP?

Quiz 2 Answers

- Boxes



Same Beliefs



Different Beliefs

- MRS is the ratio of probabilities.

Quiz 3 Answers

- For any $(\pi_1, \pi_2) > 0$, take $q_1 = \pi_1 + 2\pi_2$, $q_2 = \pi_2$ and $q_3 = 2\pi_1$. Check that $(1, \pi_1, \pi_2) \cdot M = 0$. The no-arbitrage condition implies that $q_3 = 2q_1 - 4q_2$. Any arbitrage-free asset price vector uniquely defines state prices.
- $q_1 = \pi_1 + 2\pi_3$, $q_2 = 2\pi_2 + \pi_3$. Asset prices do not uniquely define state prices. For any $\pi_3 \leq \min\{q_1/2, q_2\}$, $\pi_1 = q_1 - 2\pi_3$ and $\pi_2 = (q_2 - \pi_3)/2$.
- When the rank of A is S and there are no arbitrage possibilities.

Quiz 4 Answers

Wealth in state 1 is 1, and in state 2 is 2. The states 1 and 2 consumption bundles each cost 3/2. So 1/2 unit of wealth has to be transfered from state 2 to state 1. The state prices (Arrow security prices) are $(1, 1)$. The prices of the two assets are $q_1 = 2$ and $q_2 = 1$. The portfolio which moves 1 unit of wealth from state 2 to state 1 is $z = (-1/2, 1)$; sell a half-unit of the bond, and buy one unit of the asset that pays off only in state 1. The cost of this portfolio is, no surprise, 0.